

An Introduction to Mathematical Analysis in Economics¹

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Preface

The objective of this book is to provide a simple introduction to mathematical analysis with applications in economics. There is increasing use of real and functional analysis in economics, but few books cover that material at an elementary level. Our rationale for writing this book is to bridge the gap between basic mathematical economics books (which deal with introductory calculus and linear algebra) and advanced economics books such as Stokey and Lucas' *Recursive Methods in Economic Dynamics* that presume a working knowledge of functional analysis. The major innovations in this book relative to classic mathematics books in this area (such as Royden's *Real Analysis* or Munkres' *Topology*) are that we provide: (i) extensive simple examples (we believe strongly that examples provide the intuition necessary to grasp difficult ideas); (ii) sketches of complicated proofs (followed by the complete proof at the end of the book); and (iii) only material that is relevant to economists (which means we drop some material and add other topics (e.g. we focus extensively on set valued mappings instead of just point valued ones)). It is important to emphasize that while we aim to make this material as accessible as possible, we have not excluded demanding mathematical concepts used by economists and that the book is self-contained (i.e. virtually any theorem used in proving a given result is itself proven in our book).

Road Map

Chapter 1 is a brief introduction to logical reasoning and how to construct direct versus indirect proofs. Proving the truth of the compound statement "If A , then B " captures the essence of mathematical reasoning; we take the truth of statement " A " as given and then establish logically the truth of statement " B " follows. We do so by introducing logical connectives and the

idea of a truth table.

We introduce set operations, relations, functions and correspondences in Chapter 2 . Then we study the “size” of sets and show the differences between countable and uncountable infinite sets. Finally, we introduce the notion of an algebra (just a collection of sets that satisfy certain properties) and “generate” (i.e. establish that there always exists) a smallest collection of subsets of a given set where all results of set operations (like complements, union, and intersection) remain in the collection.

Chapter 3 focuses on the set of real numbers (denoted \mathbb{R}), which is one of the simplest but most economic (both literally and figuratively) sets to introduce students to the ideas of algebraic, order, and completeness properties. Here we expose students to the most elementary notions of distance, open and closedness, boundedness, and simple facts like between any two real numbers is another real number. One critical result we prove is the Bolzano-Weierstrass Theorem which says that every bounded infinite subset of \mathbb{R} has a point with sufficiently many points in any subset around it. This result has important implications for issues like convergence of a sequence of points which is introduced in more general metric spaces. We end by generating the smallest collection of all open sets in \mathbb{R} known as the Borel (σ -)algebra.

In Chapter 4 we introduce sequences and the notions of convergence, completeness, compactness, and connectedness in general metric spaces, where we augment an arbitrary set with an abstract notion of a “distance” function. Understanding these “C” properties are absolutely essential for economists. For instance, the completeness of a metric space is a very important property for problem solving. In particular, one can construct a sequence of approximate solutions that get closer and closer together and provided the space is complete, then the limit of this sequence *exists* and is the solution of the original problem. We also present properties of normed vector spaces and study two important examples, both of which are the used extensively in economics: finite dimensional Euclidean space (denoted \mathbb{R}^n) and the space of (infinite dimensional) sequences (denoted ℓ_p). Then we study continuity of functions and hemicontinuity of correspondences. Particular attention is paid to the properties of a continuous function on a connected domain (a generalization of the Intermediate Value Theorem) as well as a continuous function on a compact domain (a generalization of the Extreme Value Theorem). We end by providing fixed point theorems for functions and correspondences that are useful in proving, for instance, the existence of general equilibrium with competitive markets or a Nash Equilibrium of a noncooperative game.

Chapter 5 focuses primarily on Lebesgue measure and integration since almost all applications that economists study are covered by this case and because it is easy to conceptualize the notion of distance through that of the restriction of an outer measure. We show that the collection of Lebesgue measurable sets is a σ -algebra and that the collection of Borel sets is a subset of the Lebesgue measurable sets. Then we provide a set of convergence theorems for the existence of a Lebesgue integral which are applicable under a wide variety of conditions. Next we introduce general and signed measures, where we show that a signed measure can be represented simply by an integral (the Radon-Nikodym Theorem). To prepare for the following chapter, we end by studying a simple function space (the space of integrable functions) and prove it is complete.

We study properties such as completeness and compactness in two important function spaces in Chapter 6: the space of bounded continuous functions (denoted $\mathcal{C}(X)$) and the space of p -integrable functions (denoted $L_p(X)$). A fundamental result on approximating continuous functions in $\mathcal{C}(X)$ is given in a very general set of Theorems by Stone and Weierstrass. Also, the Brouwer Fixed Point Theorem of Chapter 4 on finite dimensional spaces is generalized to infinite dimensional spaces in the Schauder Fixed Point Theorem. Moving onto the $L_p(X)$ space, we show that it is complete in the Riesz-Fischer Theorem. Then we introduce linear operators and functionals, as well as the notion of a dual space. We show that one can construct bounded linear functionals on a given set X in the Hahn-Banach Theorem, which is used to prove certain separation results such as the fact that two disjoint convex sets can be separated by a linear functional. Such results are used extensively in economics; for instance, it is employed to establish the Second Welfare Theorem. The chapter ends with nonlinear operators and focuses particularly on optimization in infinite dimensional spaces. First we introduce the weak topology on a normed vector space and develop a variational method of optimizing nonlinear functions. Then we consider another method of finding the optimum of a nonlinear functional by dynamic programming.

Chapter 7 provides a brief overview of general topological spaces and the idea of a homeomorphism (i.e. when two topological spaces X and Y have “similar topological structure” which occurs when there is a one-to-one and onto mapping f from elements in X to elements in Y such that both f and its inverse are continuous). We then compare and contrast topological and metric properties, as well as touch upon the metrizable problem (i.e. finding conditions on a topological space X which guarantee that there exists

a metric on the set X that induces the topology of X).

Uses of the book

We taught this manuscript in the first year PhD core sequence at the University of Pittsburgh and as a PhD class at the University of Texas. The program at University of Pittsburgh begins with an intensive, one month remedial summer math class that focuses on calculus and linear algebra. Our manuscript was used in the Fall semester class. Since we were able to quickly explain theorems using sketches of proofs, it was possible to teach the entire book in one semester. If the book was used for upper level undergraduates, we would suggest simply to teach Chapters 1 to 4. While we used the manuscript in a classroom, we expect it will be beneficial to researchers; for instance, anyone who reads a book like Stokey and Lucas' *Recursive Methods* must understand the background concepts in our manuscript. In fact, it was because one of the authors found that his students were ill prepared to understand Stokey and Lucas in his upper level macroeconomics class, that this project began.

Chapter 1

Introduction

In this chapter we hope to introduce students to applying logical reasoning to prove the validity of economic conclusions (\mathbb{B}) from well-defined premises (\mathbb{A}). For example, \mathbb{A} may be the statement “An allocation-price pair (x, p) is a Walrasian equilibrium” and \mathbb{B} the statement “the allocation x is Pareto efficient”. In general, statements such as \mathbb{A} and/or \mathbb{B} may be true or false.

1.1 Rules of logic

In many cases, we will be interested in establishing the truth of statements of the form “If \mathbb{A} , then \mathbb{B} .” Equivalently, such a statement can be written as: “ $\mathbb{A} \Rightarrow \mathbb{B}$ ”; “ \mathbb{A} implies \mathbb{B} ”; “ \mathbb{A} only if \mathbb{B} ”; “ \mathbb{A} is sufficient for \mathbb{B} ”; or “ \mathbb{B} is necessary for \mathbb{A} .” Applied to the example given in the previous paragraph, “If \mathbb{A} , then \mathbb{B} ” is just a statement of the First Fundamental Theorem of Welfare Economics. In other cases, we will be interested in the truth of statements of the form “ \mathbb{A} if and only if \mathbb{B} .” Equivalently, such a statement can be written: “ $\mathbb{A} \Rightarrow \mathbb{B}$ and $\mathbb{B} \Rightarrow \mathbb{A}$ ” which is just “ $\mathbb{A} \Leftrightarrow \mathbb{B}$ ”; “ \mathbb{A} implies \mathbb{B} and \mathbb{B} implies \mathbb{A} ”; “ \mathbb{A} is necessary and sufficient for \mathbb{B} ”; or “ \mathbb{A} is equivalent to \mathbb{B} .”

Notice that a statement of the form “ $\mathbb{A} \Rightarrow \mathbb{B}$ ” is simply a construct of two simple statements connected by “ \Rightarrow ”. Proving the truth of the statement “ $\mathbb{A} \Rightarrow \mathbb{B}$ ” captures the essence of mathematical reasoning; we take the truth of \mathbb{A} as given and then establish logically the truth of \mathbb{B} follows. Before actually setting out on that path, let us define a few terms. A *Theorem* or *Proposition* is a statement that we prove to be true. A *Lemma* is a theorem we use to prove another theorem. A *Corollary* is a theorem whose proof is

Since every case in which $\mathbb{A} \vee (\mathbb{B} \wedge \mathbb{C})$ is true or false, so is $(\mathbb{A} \vee \mathbb{B}) \wedge (\mathbb{A} \vee \mathbb{C})$, the two statements are equivalent.

Theorem 1 *Let \mathbb{A} , \mathbb{B} , and \mathbb{C} be any statements. Then*

$$(\mathbb{A} \vee \mathbb{B}) \Leftrightarrow (\mathbb{B} \vee \mathbb{A}) \text{ and } (\mathbb{A} \wedge \mathbb{B}) \Leftrightarrow (\mathbb{B} \wedge \mathbb{A}) \quad (1.1)$$

$$((\mathbb{A} \vee \mathbb{B}) \vee \mathbb{C}) \Leftrightarrow (\mathbb{A} \vee (\mathbb{B} \vee \mathbb{C})) \text{ and } ((\mathbb{A} \wedge \mathbb{B}) \wedge \mathbb{C}) \Leftrightarrow (\mathbb{A} \wedge (\mathbb{B} \wedge \mathbb{C})) \quad (1.2)$$

$$(\mathbb{A} \vee (\mathbb{B} \wedge \mathbb{C})) \Leftrightarrow ((\mathbb{A} \vee \mathbb{B}) \wedge (\mathbb{A} \vee \mathbb{C})) \text{ and } (\mathbb{A} \wedge (\mathbb{B} \vee \mathbb{C})) \Leftrightarrow (\mathbb{A} \wedge \mathbb{B}) \vee (\mathbb{A} \wedge \mathbb{C}) \quad (1.3)$$

Exercise 1.1.1 *Complete the proof of Theorem 1.*

The next set of results form the basis of the methods of logical reasoning we will be pursuing in this book. The first (direct) approach (1.4) is the *syllogism*, which says that “if \mathbb{A} is true and \mathbb{A} implies \mathbb{B} , then \mathbb{B} is true”. The second (indirect) approach (1.5) is the *contradiction*, which says in words that “if not \mathbb{A} leads to a false statement of the form \mathbb{B} and not \mathbb{B} , then \mathbb{A} is true. That is, one way to prove \mathbb{A} is to hypothesize $\sim \mathbb{A}$, and show this leads to a contradiction. Another (indirect) approach (1.6) is the *contrapositive*, which says that “ \mathbb{A} implies \mathbb{B} is the same as whenever \mathbb{B} is false, \mathbb{A} is false”.

Theorem 2

$$(\mathbb{A} \wedge (\mathbb{A} \Rightarrow \mathbb{B})) \Rightarrow \mathbb{B} \quad (1.4)$$

$$((\sim \mathbb{A}) \Rightarrow (\mathbb{B} \wedge (\sim \mathbb{B}))) \Rightarrow \mathbb{A} \quad (1.5)$$

$$(\mathbb{A} \Rightarrow \mathbb{B}) \Leftrightarrow ((\sim \mathbb{B}) \Rightarrow (\sim \mathbb{A})). \quad (1.6)$$

Proof. Before proceeding, we need a few results (we could have established these in the form of a lemma, but we’re just starting here). The first result¹ we need is that

$$(\mathbb{A} \Rightarrow \mathbb{B}) \Leftrightarrow ((\sim \mathbb{A}) \vee \mathbb{B}) \quad (1.7)$$

and the second is

$$\sim (\sim \mathbb{A}) \Leftrightarrow \mathbb{A}. \quad (1.8)$$

¹The result follows from

\mathbb{A}	\mathbb{B}	$\mathbb{A} \Rightarrow \mathbb{B}$	$\sim \mathbb{A} \vee \mathbb{B}$
T	T	T	T

In the case of (1.4), $(\mathbb{A} \wedge (\mathbb{A} \Rightarrow \mathbb{B})) \stackrel{(1.7)}{\Leftrightarrow} (\mathbb{A} \wedge ((\sim \mathbb{A}) \vee \mathbb{B})) \stackrel{(1.3)}{\Leftrightarrow} (\mathbb{A} \wedge (\sim \mathbb{A})) \vee (\mathbb{A} \wedge \mathbb{B}) \Rightarrow \mathbb{B}$ by table 1.1.

In the case of (1.5), $((\sim \mathbb{A}) \Rightarrow (\mathbb{B} \wedge (\sim \mathbb{B}))) \stackrel{(1.7)}{\Leftrightarrow} (\mathbb{A} \vee (\mathbb{B} \wedge (\sim \mathbb{B}))) \Rightarrow \mathbb{A}$ by table 1.1.

In the case of (1.6), $(\mathbb{A} \Rightarrow \mathbb{B}) \stackrel{(1.7)}{\Leftrightarrow} ((\sim \mathbb{A}) \vee \mathbb{B}) \stackrel{(1.1)}{\Leftrightarrow} (\mathbb{B} \vee (\sim \mathbb{A})) \stackrel{(1.8)}{\Leftrightarrow} (\sim (\sim \mathbb{B}) \vee (\sim \mathbb{A})) \stackrel{(1.7)}{\Leftrightarrow} ((\sim \mathbb{B}) \Rightarrow (\sim \mathbb{A})). \blacksquare$

Note that the contrapositive of “ $\mathbb{A} \Rightarrow \mathbb{B}$ ” is not the same as the *converse* of “ $\mathbb{A} \Rightarrow \mathbb{B}$ ”, which is “ $\mathbb{B} \Rightarrow \mathbb{A}$ ”.

Another important way to “construct” complicated statements from simple ones is by the use of *quantifiers*. In particular, a quantifier allows a statement $\mathbb{A}(x)$ to vary across elements x in some universe U . For example, x could be a price (whose universe is always positive) with the property that demand equals supply. When there is an x with the property $\mathbb{A}(x)$, we write $(\exists x)\mathbb{A}(x)$ to mean that for some x in U , $\mathbb{A}(x)$ is true.² In the context of the previous example, this establishes there exists an equilibrium price. When all x have the property $\mathbb{A}(x)$, we write $(\forall x)\mathbb{A}(x)$ to mean that for all x , $\mathbb{A}(x)$ is true.³ There are obvious relations between “ \exists ” and “ \forall ”. In particular

$$\sim ((\exists x)\mathbb{A}(x)) \Leftrightarrow (\forall x)(\sim \mathbb{A}(x)) \quad (1.9)$$

$$\sim ((\forall x)\mathbb{A}(x)) \Leftrightarrow (\exists x)(\sim \mathbb{A}(x)). \quad (1.10)$$

The second tautology is important since it illustrates the concept of a *counterexample*. In particular, (1.10) states “If it is not true that $\mathbb{A}(x)$ is true for all x , then there must exist a counterexample (that is, an x satisfying $\sim \mathbb{A}(x)$), and vice versa. Counterexamples are an important tool, since while hundreds of examples do not make a theorem, a single counterexample kills one.

One should also note that the symmetry we experienced with “ \vee ” and “ \wedge ” in (1.1) to (1.3) may break down with quantifiers. Thus while

$$(\exists x)(\mathbb{A}(x) \vee \mathbb{B}(x)) \Leftrightarrow (\exists(x)\mathbb{A}(x) \vee \exists(x)\mathbb{B}(x)) \quad (1.11)$$

can be expressed as a tautology (i.e. “ \Leftrightarrow ”), it’s the case that

$$(\exists x)(\mathbb{A}(x) \wedge \mathbb{B}(x)) \Rightarrow (\exists(x)\mathbb{A}(x) \wedge \exists(x)\mathbb{B}(x)) \quad (1.12)$$

²Thus, we let “ \exists ” denote “for some” or “there exists a”.

³Thus, we let “ \forall ” denote “for all”.

cannot be expressed that way (i.e. it is only " \Rightarrow "). To see why (1.12) cannot hold as an "if and only if" statement, suppose x is the set of countries in the world, $\mathbb{A}(x)$ is the property that x is above average gross domestic product and $\mathbb{B}(x)$ is the property that x is below average gross domestic product, then there will be at least one country above the mean and at least one country below the mean (i.e. $(\exists(x)\mathbb{A}(x) \wedge \exists(x)\mathbb{B}(x))$ is true), but clearly there cannot be a country that is both above and below the mean (i.e. $(\exists(x) (\mathbb{A}(x) \wedge \mathbb{B}(x)))$ is false).

We can make increasingly complex statements by adding more variables (e.g. the statement $\mathbb{A}(x, y)$ can vary across elements x and y in some universe \tilde{U}). For instance, when $\mathbb{A}(x, y)$ states that " y that is larger than x " where x and y are in the universe of real numbers, the statement $(\forall x)(\exists y)(x < y)$ says "for every x there is a y that is larger than x ", while the statement $(\exists y)(\forall x)(x < y)$ says "there is a y which is larger than every x ". Note, however, the former statement is true, but the latter is false.

1.2 Taxonomy of Proofs

While the previous section introduced the basics of the rules of logic (how to manipulate connectives and quantifiers to establish the truth of statements), here we will discuss broadly the methodology of proofs you will frequently encounter in economics. The most intuitive is the *direct proof* in the form of " $\mathbb{A} \Rightarrow \mathbb{B}$ ", discussed in (1.4). The work is to fill in the intermediate steps so that $\mathbb{A} \Rightarrow \mathbb{A}_1$ and $\mathbb{A}_1 \Rightarrow \mathbb{A}_2$ and ... $\mathbb{A}_{n-1} \Rightarrow \mathbb{B}$ are all tautologies.

In some cases, it may be simpler to prove a statement like $\mathbb{A} \Rightarrow \mathbb{B}$ by splitting \mathbb{B} into cases. For example, if we wish to prove the uniqueness of the least upper bound of a set $A \subset \mathbb{R}$, we can consider two candidate least upper bounds x_1 and x_2 in A and split \mathbb{B} into the cases where we assume x_1 is the least upper bound implying $x_1 \leq x_2$ and another case where we assume x_2 is the least upper bound implying $x_2 \leq x_1$. But $(x_1 \leq x_2) \wedge (x_2 \leq x_1) \Rightarrow (x_1 = x_2)$ so that the least upper bound is unique. In other instances, one might want to split \mathbb{A} into cases (call them \mathbb{A}^1 and \mathbb{A}^2), show $\mathbb{A} \Leftrightarrow (\mathbb{A}^1 \vee \mathbb{A}^2)$ and then show $\mathbb{A}^1 \Rightarrow \mathbb{A}$ and $\mathbb{A}^2 \Rightarrow \mathbb{A}$. For example, to prove

$$(0 \leq x \leq 1) \Rightarrow (x^2 \leq x)$$

we can use the fact that

$$(0 \leq x \leq 1) \Leftrightarrow (x = 0 \vee (0 < x \leq 1))$$

where the latter case allows us to consider the truth of \mathbb{B} by dividing through by x .

Another direct method of proof, called *induction*, works only for the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. Suppose we wish to show $(\forall n \in \mathbb{N}) \mathbb{A}(n)$ is true. This is equivalent to proving $\mathbb{A}(0) \wedge (\forall n \in \mathbb{N}) (\mathbb{A}(n) \Rightarrow \mathbb{A}(n+1))$. This works since $\mathbb{A}(0)$ is true and $\mathbb{A}(0) \Rightarrow \mathbb{A}(1)$ and $\mathbb{A}(1) \Rightarrow \mathbb{A}(2)$ and so on. In the next chapter, after we introduce set theory, we will show why induction works.

As discussed before, two indirect forms of proof are the *contrapositive* (1.6) and the *contradiction* (1.5). In the latter case, we use the fact that $\sim (\mathbb{A} \Rightarrow \mathbb{B}) \Leftrightarrow (\mathbb{A} \wedge (\sim \mathbb{B}))$ and show $(\mathbb{A} \wedge (\sim \mathbb{B}))$ leads to a contradiction $(\mathbb{B} \wedge (\sim \mathbb{B}))$. Since direct proofs seem more natural than indirect proofs, we now give an indirect proof of the First Welfare Theorem, perhaps one of the most important things you will learn in all of economics. It is so simple, that it is hard to find a direct counterpart.⁴

Definition 3 Given a finite vector of endowments y , an allocation x is **feasible** if for each good k ,

$$\sum_i x_{i,k} \leq \sum_i y_{i,k} \quad (1.13)$$

where the summation is over all individuals in the economy.

Definition 4 A feasible allocation x is a **Pareto efficient** allocation if there is no feasible allocation x' such that all agents prefer x' to x .

Definition 5 An allocation-price pair (x, p) in a competitive exchange economy is a **Walrasian equilibrium** if it is feasible and if x'_i is preferred by i to x_i , then each agent i is maximized in his budget set

$$\sum_k p_k x'_{i,k} > \sum_k p_k y_{i,k} \quad (1.14)$$

(i.e. i 's tastes outweigh his pocketbook).

Theorem 6 (First Fundamental Theorem of Welfare Economics) If (x, p) is a Walrasian equilibrium, then x is Pareto efficient.

⁴See Debreu (1959, p.94).

Proof. By contradiction. Suppose x is not Pareto efficient. Let x' be a feasible allocation that all agents prefer to x . Then by the definition of Walrasian equilibrium, we can sum (1.14) across all individuals to obtain

$$\sum_i \left(\sum_k p_k x'_{i,k} \right) > \sum_i \left(\sum_k p_k y_{i,k} \right) \Leftrightarrow \sum_k p_k \left(\sum_i x'_{i,k} \right) > \sum_k p_k \left(\sum_i y_{i,k} \right). \quad (1.15)$$

Since x' is a feasible allocation, summing (1.13) over all goods we have

$$\sum_k \sum_i p_k x'_{i,k} \leq \sum_k \sum_i p_k y_{i,k}. \quad (1.16)$$

But (1.15) and (1.16) imply

$$\sum_k \sum_i p_k y_{i,k} > \sum_k \sum_i p_k y_{i,k},$$

which is a contradiction. ■

Here \mathbb{B} is the statement "x is Pareto Efficient". So the proof by contradiction assumes $\sim \mathbb{B}$, which is "Suppose x is not Pareto Efficient". In that case, by definition 4, there's a preferred allocation x' which is feasible. But if x' is preferred to x , then it must cost too much if it wasn't chosen in the first place (this is 1.14). But this contradicts that x' was feasible.

1.3 Bibliography for Chapter 1

An excellent treatment of this material is in McAfee (1986, Economics 241 handout). See also Munkres (1975, p. 7-9) and Royden (1988, p. 2-3).

Chapter 2

Set Theory

The basic notions of set theory are those of a group of objects and the idea of membership in that group. In what follows, we will *fix* a given universe (or space) X and consider only sets (or groups) whose elements (or members) are elements of X . We can express the notion of membership by “ \in ” so that “ $x \in A$ ” means “ x is an element of the set A ” and “ $x \notin A$ ” means “ x is not an element of A ”. Since a set is completely determined by its elements, we usually specify its elements explicitly by saying “The set A is the set of all elements x in X such that each x has the property \mathbb{A} (i.e. that $\mathbb{A}(x)$ is true)” and write $A = \{x \in X : \mathbb{A}(x)\}$.¹ This also makes it clear that we identify sets with statements.

Example 7 *Agent i 's budget set, denoted $B_i(p, y_i) = \{x_i \in X : \sum_k p_k x_{i,k} \leq \sum_k p_k y_{i,k}\}$, is the set of all consumption goods that can be purchased with endowments y_i .*

Definition 8 *If each $x \in A$ is also in the set B (i.e. $x \in A \Rightarrow x \in B$), then we say A is a **subset** of B (denoted $A \subset B$). If $A \subset B$ and $\exists x \in B$ such that $x \notin A$, then A is a **proper subset** of B . If $A \subset B$, then it is equivalent to say that B **contains** A (denoted $B \supset A$).*

Definition 9 *A **collection** is a set whose elements are subsets of X . The **power set** of X , denoted $\mathcal{P}(X)$, is the set of all possible subsets of X (it has $2^{\#(X)}$ elements, where $\#(X)$ denotes the number of elements (or cardinality)*

¹In those instances where the space is understood, we sometimes abbreviate this as $A = \{x : \mathbb{A}(x)\}$.

of the set X). A **family** is a set whose elements are collections of subsets of X .

Definition 10 Two sets are **equal** if $(A \subset B) \wedge (B \subset A)$ (denoted $A = B$).

Definition 11 A set that has no elements is called **empty** (denoted \emptyset). Thus, $\emptyset = \{x : x \in X : \mathbb{A}(x) \wedge (\sim \mathbb{A}(x))\}$.

The empty set serves the same role in the theory of sets as 0 serves in the counting numbers; it is a placeholder.

Example 12 Let the universe be given by $X = \{a, b, c\}$. We could let $A = \{a, b\}$, $B = \{c\}$ be subsets of X , $\mathcal{C} = \{A, B\}$, $\mathcal{D} = \{\emptyset\}$, $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ be collections, and $\mathcal{F} = \{\mathcal{C}\}$ be a family.

The next result provides the first example of the relation between set theory and logical rules we developed in Chapter 1. In particular, it relates " \subset " and " \Rightarrow " as well as " $=$ " and " \Leftrightarrow ".

Theorem 13 Let $A = \{x \in X : \mathbb{A}(x)\}$ and $B = \{x \in X : \mathbb{B}(x)\}$. Then (a) $A \subset B \Leftrightarrow (\forall x \in X)(\mathbb{A}(x) \Rightarrow \mathbb{B}(x))$ and (b) $A = B \Leftrightarrow (\forall x \in X)(\mathbb{A}(x) \Leftrightarrow \mathbb{B}(x))$.

Proof. Just use definition (8) in (a) $A \subset B \Leftrightarrow x \in A \Rightarrow x \in B \Leftrightarrow \mathbb{A}(x) \Rightarrow \mathbb{B}(x)$ and definition (10) in (b) $A = B \Leftrightarrow (A \subset B) \wedge (B \subset A) \Leftrightarrow (\forall x \in X)(\mathbb{A}(x) \Leftrightarrow \mathbb{B}(x))$. ■

The following are some of the most important sets we will encounter in this book:

- $\mathbb{N} = \{1, 2, 3, \dots\}$, the natural or "counting" numbers.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the integers. $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, the non-negative integers.
- $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$, the rational numbers.
- Chapter 3 will discuss the real numbers, which we denote \mathbb{R} . This set just adds what are called irrational numbers to the above rationals.

There are several important results you will see at the end of this chapter. The first establishes that there are fundamentally different sizes of infinite sets. While some infinite sets can be counted, others are uncountable. These results are summarized in Theorem 71 and Theorem 80. The second result establishes that there always exists a smallest collection of subsets of a given set where all results of set operations (like complements, union, and intersection) remain in the collection (Theorem 87).

2.1 Set Operations

The following operations help us construct new sets from old ones. The first three play the same role for sets as the connectives “ \sim ”, “ \wedge ”, and “ \vee ” played for statements.

Definition 14 If $A \subset X$, we define the **complement of A (relative to X)** (denoted A^c) to be the set of all elements of X that do not belong to A . That is, $A^c = \{x \in X : x \notin A\}$.

Definition 15 If $A, B \subset X$, we define their **intersection** (denoted $A \cap B$) to be the set of all elements that belong to both A and B . That is, $A \cap B = \{x \in X : x \in A \wedge x \in B\}$.

Definition 16 If $A, B \subset X$ and $A \cap B = \emptyset$, then we say A and B are **disjoint**.

Definition 17 If $A, B \subset X$, we define their **union** (denoted $A \cup B$) to be the set of all elements that belong to A or B or both (i.e. or is inclusive). That is, $A \cup B = \{x \in X : x \in A \vee x \in B\}$.

Definition 18 If $A, B \subset X$, we define their **difference (or relative complement of A in B)** (denoted $A \setminus B$) to be the set of all elements of A that do not belong to B . That is, $A \setminus B = \{x \in X : x \in A \wedge x \notin B\}$.

Each of these definitions can be visualized in Figure 2.1.1 through the use of Venn Diagrams. These definitions can easily be extended to arbitrary collections of sets. Let Λ be an index set (e.g. $\Lambda = \mathbb{N}$ or a finite subset of \mathbb{N}) and let $A_i, i \in \Lambda$ be subsets of X . Then $\cup_{i \in \Lambda} A_i = \{x \in X : (\exists i)(x \in A_i)\}$. Indexed families of sets will be defined formally after we develop the notion of a function in Section 5.2.

2.1.1 Algebraic properties of set operations

The following commutative, associative, and distributive properties of sets are natural extensions of Theorem 1 and easily seen in Figure 2.1.2.

Theorem 19 *Let A, B, C be any sets. Then (i) (C) $A \cap B = B \cap A$, $A \cup B = B \cup A$; (ii) (A) $(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$; and (iii) (D) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

Exercise 2.1.1 *Prove Theorem 19. This amounts to applying the logical connectives and above definitions. Besides using Venn Diagrams, we can just use the definition of \cap and \cup . For example, to show $A \cap B = B \cap A$, it is sufficient to note $x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B) \stackrel{1,1}{\Leftrightarrow} (x \in B) \wedge (x \in A) \Leftrightarrow x \in B \cap A$.*

The following properties are used extensively in probability theory and are easily seen in Figure 2.1.3.

Theorem 20 *(DeMorgan's Laws) If A, B, C are any sets, then (a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$, and (b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.*

Proof. (a) 2 parts.

(i, \Rightarrow) Suppose $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin (B \cup C)$. Thus $x \in A$ and $(x \notin B \text{ and } x \notin C)$. This implies $x \in A \setminus B$ and $x \in A \setminus C$. But this is just $x \in (A \setminus B) \cap (A \setminus C)$.

(ii, \Leftarrow) Suppose $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in (A \setminus B)$ and $x \in (A \setminus C)$. Thus $x \in A$ and $(x \notin B \text{ or } x \notin C)$. This implies $x \in A$ and $x \notin (B \cup C)$. But this is just $x \in A \setminus (B \cup C)$. ■

Exercise 2.1.2 *Finish the proof of Theorem 20.*

2.2 Cartesian Products

There is another way to construct new sets out of given ones; it involves the notion of an “ordered pair” of objects. That is, in the set $\{a, b\}$ there is no preference given to a over b ; i.e. $\{a, b\} = \{b, a\}$ so that it is an unordered pair. We can also consider ordered pairs (a, b) where we distinguish between the first and second elements.²

²Don't confuse this notation with the interval consisting of all real numbers such that $a < x < b$.

Definition 21 If A and B are nonempty sets, then the **cartesian product** (denoted $A \times B$) is just the set of all ordered pairs $\{(a, b) : a \in A \text{ and } b \in B\}$.

Example 22 $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$.

Example 23 $A = [0, 1] \cup [2, 3]$, $B = [1, 2] \cup [3, 4]$, $A \times B$ in Figure 2.2.1

This set operation also generalizes to finite and infinite index sets.

2.3 Relations

To be able to compare elements of a set, we need to define how they are related. The general concept of a relation underlies all that will follow. For instance, just comparing the real numbers 1 and 2 requires such a definition. Furthermore, a correspondence or function is just a special case of a relation. In what follows, our definitions of relations, correspondences, and functions are meant to emphasize that they are simply special kinds of sets.

Definition 24 Given two sets A and B , a **binary relation** between members of A and members of B is a subset $R \subset A \times B$. We use the notation $(a, b) \in R$ to denote the relation R on $A \times B$ and read it “ a is in the relation R to b ”. If $A = B$ we say that R is the **relation** on the set A .

Example 25 Let $A = \{\text{Austin, Des Moines, Harrisburg}\}$ and $B = \{\text{Texas, Iowa, Pennsylvania}\}$. Then the relation $R = \{(\text{Austin, Texas}), (\text{Des Moines, Iowa}), (\text{Harrisburg, Pennsylvania})\}$ expresses “is the state capital of”.

In general, we can consider **n-nary relations** between members of sets A_1, A_2, \dots, A_n which is just the subset $R \subset A_1 \times A_2 \times \dots \times A_n$.

A relation is characterized by a certain set of properties that it possesses. We next consider important types of relations that differ in their symmetry properties.

2.3.1 Equivalence relations

Definition 26 An **equivalence relation** on a set A is a relation ‘ \sim ’ having the following three properties: (i) Reflexivity, $x \sim x$, $\forall x \in A$; (ii) Symmetry, if $x \sim y$, then $y \sim x$, $\forall x, y \in A$; and (iii) Transitivity, if $x \sim y$ and $y \sim z$, then $x \sim z$, $\forall x, y, z \in A$.

Example 27 Equality is an equivalence relation on \mathbb{R} .

Example 28 Define the **congruence modulo 4** relation ' M ' on \mathbb{Z} by $\forall x, y \in \mathbb{Z}, xMy$ if remainders obtained by dividing x and y by 4 are equal. For example, $13M65$ because dividing 13 and 65 by 4 give the same remainder of 1.

Exercise 2.3.1 Show that congruence modulo 4 is an equivalence relation.

Definition 29 Given an equivalence relation \sim on a set A and an element $x \in A$, we define a certain subset E of A called the **equivalence class determined by x** by the equation $E = \{y \in A : y \sim x\}$. Note that the equivalence class determined by x contains x since $x \sim x$.

Example 30 The equivalence classes of \mathbb{Z} for the relation congruence modulo 4 are determined by $x \in \{0, 1, 2, 3\}$ where $E_x = \{z \in \mathbb{Z} : z = 4k + x, k \in \mathbb{Z}\}$ (i.e. x is the remainder when z is divided by 4).

Equivalence classes have the following property.

Theorem 31 Two equivalence classes E and E' are either disjoint or equal.

Proof. Let $E = \{y \in A : y \sim x\}$ and $E' = \{y \in A : y \sim x'\}$. Consider $E \cap E'$. It can be either empty (in which case E and E' are disjoint) or nonempty. Let $z \in E \cap E'$. We show that $E = E'$. Let $w \in E$. Then $w \sim x$. Since $z \in E \cap E'$, we know $z \sim x$ and $z \sim x'$ so that by transitivity $x \sim x'$. Also by transitivity $w \sim x'$ so that $w \in E'$. Thus $E \subset E'$. Symmetry allows us to conclude that $E' \subset E$ as well. Hence $E = E'$. ■

Given an equivalence relation on A , let us denote by \mathcal{E} the collection of all equivalence classes. Theorem 31 shows that distinct elements of \mathcal{E} are disjoint. On the other hand, the union of all the elements of \mathcal{E} equals all of A because every element of A belongs to an equivalence class. In this case we say that \mathcal{E} is a partition of A .

Definition 32 A **partition** of a set A is a collection of disjoint subsets of A whose union is all of A .

Example 33 It is clear that the equivalence classes of \mathbb{Z} in Example (30) is a partition since, for instance, $E_0 = \{\dots, -8, -4, 0, 4, 8, \dots\}$, $E_1 = \{\dots, -5, -1, 1, 5, \dots\}$, $E_2 = \{\dots, -6, -2, 2, 6, \dots\}$, $E_3 = \{\dots, -7, -3, 3, 7, \dots\}$ are disjoint and their union is all of \mathbb{Z} . Another simple example is a coin toss experiment where the sample space $S = \{\text{Heads}, \text{Tails}\}$ has mutually exclusive events (i.e. $\text{Heads} \cap \text{Tails} = \emptyset$).

2.3.2 Order relations

A relation that is reflexive and transitive but not symmetric is said to be an *order relation*. If we consider special types of non symmetry, we have special types of order relations.

Definition 34 A relation ‘ R ’ on A is said to be a **partial ordering** of a set A if it has the following properties: (i) Reflexivity, $xRx, \forall x \in A$; (ii) Antisymmetry, if xRy and yRx , then $x = y, \forall x, y \in A$; and (iii) Transitivity, if xRy and yRz , then $xRz, \forall x, y, z \in A$. We call (R, A) a **partially ordered set**.

Example 35 ‘ \leq ’ is a partial ordering on \mathbb{R} and ‘ \subset ’ is a partial ordering on $\mathcal{P}(A)$. It is clear that \leq is not symmetric on \mathbb{R} ; just take $x = 1$ and $y = 2$. It is also clear that \subset is not symmetric on $\mathcal{P}(A)$; if $A = \{a, b\}$, then while $\{a\} \subset A$ it is not the case that $A \subset \{a\}$. Finally, ‘ \lesssim^1 ’ on $\mathbb{R} \times \mathbb{R}$ given by $(x_1, x_2) \lesssim^1 (y_1, y_2)$ if $x_1 \leq y_1$ and $x_2 \leq y_2$ is a partial ordering since it is clear that \lesssim^1 is not symmetric on $\mathbb{R} \times \mathbb{R}$ because \leq is not symmetric even on \mathbb{R} .

Definition 36 A partially ordered relation ‘ R ’ on A is said to be a **total (or linear) ordering** of A if (i) Completeness, for any two elements $x, y \in A$ we have either xRy or yRx . We call (R, A) a **totally ordered set**. A **chain** in a partially ordered set is a subset on which the order is total.

Thus, a total ordering means that any two elements x and y in A can be compared, unlike a partial ordering where there are elements that are noncomparable.

Exercise 2.3.2 Show that if $A \subset B$ and B is totally ordered, then A is totally ordered.

We write $x \prec y$ if $x \preceq y$ and $x \neq y$, and call ‘ \prec ’ a *strict partial* or *strict total ordering*.

Example 37 ‘ $<$ ’ is a strict total ordering on \mathbb{R} while ‘ \leq ’ is a total ordering on \mathbb{R} , both of which follow by the completeness axiom of real numbers. ‘ \subset ’ is not a total ordering on $\mathcal{P}(A)$ since if $A = \{a, b\}$, there is no inclusion relation between the sets $\{a\}$ and $\{b\}$. ‘ \lesssim^1 ’ on $\mathbb{R} \times \mathbb{R}$ given in Example 35 is not a

total ordering because we can't compare elements where $x_1 \leq y_1$ and $x_2 \geq y_2$. However, a line passing through the origin having positive slope is a chain. On the other hand, the relation ' \succsim^2 ' on $\mathbb{R} \times \mathbb{R}$ given by $(x_1, x_2) \succsim^2 (y_1, y_2)$ if $x_1 \leq y_1$ or if $x_1 = y_1$ and $x_2 \leq y_2$ is a total ordering.³ This is also known as a lexicographic ordering since the first element of the totally ordered set has the highest priority in determining the ordering just as the first letter of a word does in the ordering of a dictionary. We compare ' \succsim^1 ' to ' \succsim^2 ' in Figure 2.3.1 for the following four elements $x = (\frac{1}{4}, \frac{1}{4})$, $y = (\frac{1}{2}, \frac{1}{2})$, $z = (\frac{1}{4}, \frac{3}{4})$ in $\mathbb{R} \times \mathbb{R}$. There are 3 pairwise comparisons for each relation. First consider ' \succsim^1 '. We have $x \succsim^1 y$, $x \succsim^1 z$ but y and z are not comparable under ' \succsim^1 ', which is why we call it a partial ordering. Next consider ' \succsim^2 ', where each pair is comparable (i.e. we have $x \succsim^2 z$, $x \succsim^2 y$, and $z \succsim^2 y$) which is why we call it a total ordering. Notice that by transitivity they can be ranked (all can be placed in the dictionary).

There are other types of order relations.

Definition 38 A weak order relation assumes: (i) transitivity; (ii) completeness; and (iii) non symmetry (just the negation of symmetry defined in 26).⁴

Weak order relations form the basis for consumer choice.

Example 39 Preference relations: We can represent consumer preferences by the binary relation \succsim defined on a non-empty, closed, convex consumption set X . If $(x^1, x^2) \in \succsim$ or $x^1 \succsim x^2$ we say "consumption bundle x^1 is at least as good as x^2 ". We embody rationality or consistency by completeness and transitivity.⁵

Exercise 2.3.3 Why aren't preference relations just total orderings? Why are they weak orderings? Show why indifference is an equivalence relation.

Because elements of a partially ordered set are not necessarily comparable, it may be the case that a maximum and/or minimum of a two element set doesn't even exist. We turn to this next.

³Don't be confused that we have left out a case (i.e. $x_1 > y_1$) by considering only $x_1 \leq y_1$ or if $x_1 = y_1$ and $x_2 \leq y_2$. For instance, if the two elements we are considering are (2, 3) and (1, 7), simply take $x = (1, 7)$ and $y = (2, 3)$. The point is that any two real numbers can be compared using ' \leq '.

⁴Reflexivity is implied by completeness.

⁵Experiments show that transitivity is often violated.

Definition 40 Let \lesssim be a partial ordering of X . An **upper bound** for a set $A \subset X$ is an element $u \in X$ satisfying $x \lesssim u$, $\forall x \in A$. The **supremum** of a set is its least upper bound and when the set contains its supremum we call it a **maximum**. A **lower bound** for a set $A \subset X$ is an element $l \in X$ satisfying $l \lesssim x$, $\forall x \in A$. The **infimum** of a set is its greatest lower bound and when the set contains its infimum we call it a **minimum**.

Definition 41 A set S is **bounded above** if it has an upper bound; **bounded below** if it has a lower bound; **bounded** if it has an upper and lower bound; **unbounded** if it lacks either an upper or a lower bound.

We define the operators $x \vee y$ to denote the supremum and $x \wedge y$ the infimum of the two point set $\{x, y\}$.⁶ If X is a total order, then x and y are comparable, so that one must be bigger or smaller than the other in which case $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. However, if X is a partial order, then x and y may not be comparable but we can still find their supremum and infimum.

Definition 42 A **lattice** is a partially ordered set in which every pair of elements has a supremum and an infimum.

Exercise 2.3.4 Show that: (i) every finite set in a lattice has a supremum and an infimum; and (ii) if a lattice is totally ordered, then every pair of elements has a minimum and a maximum. Hint: (i) $\sup\{x_1, x_2, x_3\} = \sup\{\sup\{x_1, x_2\}, x_3\}$.

Exercise 2.3.5 Show that a totally ordered set L is always a lattice.

Next we give examples of partially ordered sets that are not totally ordered yet have a lattice structure. For any set X , an example is $\mathcal{P}(X)$ with \subset is a lattice where if $A, B \in \mathcal{P}(X)$, then $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

Example 43 Let $X = \{a, b\}$, so that $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then, for instance, $\{a\} \vee \{b\} = \{a, b\}$, $\{a\} \wedge \{b\} = \emptyset$, $\{a\} \wedge \{a, b\} = \{a\}$, and $\{a\} \vee \emptyset = \{a\}$.

⁶Here's another place where we don't have enough good symbols to go around. Don't confuse ' \vee ' and ' \wedge ' here with the logical connectives in Chapter 1.

Example 44 $\mathbb{R} \times \mathbb{R}$ is a lattice with the ordering ' \preceq^1 '. The infimum and supremum of any two points x, y are given by $x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\})$ and $x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\})$. See Figure 2.3.2 where we consider the noncomparable elements $x = (1, 0)$ and $y = (0, 1)$.

Example 45 The next example shows that not every partially ordered set is a lattice. We show this by resorting to the following subset $X = \{(x_1, x_2) \in \mathbb{R} : x_1^2 + x_2^2 \leq 1\}$. For ' \preceq^1 ' on X , $\sup\{(0, 1), (1, 0)\}$ does not exist. See Figure 2.3.3.

While the next result is stated as a lemma, we will take it as an axiom.⁷ It will prove useful in separation theorems which are used extensively in economics.

Lemma 46 (Zorn) If A is a partially ordered set such that each totally ordered subset (a chain) has an upper bound in A , then A has a maximal element.

Example 47 $(1, 1)$ is the maximal element of ' \preceq^1 ' on $A = [0, 1] \times [0, 1]$. The upper bounds of each chain in A are given by the intersection of the lines (chains) with the $x = 1$ or $y = 1$ axes. See Figure 2.3.4.

ADD WELL ORDERING???

2.4 Correspondences and Functions

In your first economics classes you probably saw downward sloping demand and upward sloping supply functions, and perhaps even correspondences (e.g. backward bending labor supply curves). Given that we have already introduced the idea of a relation, here we will define correspondences and functions simply as a relation which has certain properties.

Definition 48 Let A and B be any two sets. A **correspondence** G , denoted $G : A \rightarrow B$, is a relation between A and $\mathcal{P}(B)$ (i.e. $G \subset A \times \mathcal{P}(B)$). That is, G is a rule that assigns a subset $G(a) \subset B$ to each element $a \in A$.

⁷It can be shown to be equivalent to the Axiom of choice.

Definition 49 Let A and B be any two sets. A **function (or mapping)** f , denoted $f : A \rightarrow B$, is a relation between A and B (i.e. $f \subset A \times B$) satisfying the following property: if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$. That is, f is a rule that assigns a unique element $f(a) \in B$ to each $a \in A$. A is called the **domain** of f , sometimes denoted $D(f)$. The **range** of f , denoted $R(f)$, is $\{b \in B : \exists a \in A \text{ such that } (a, b) \in f\}$. The **graph** of f is $G(f) = \{(a, b) \in f : \forall a \in A\}$.

Thus, a function can be thought of as a single valued correspondence. A function is defined if the following is given: (i) The domain $D(f)$. (ii) An assignment rule $a \rightarrow f(a) = b$, $a \in D(f)$. Then $R(f)$ is determined by these two. See Figure 2.4.1a for a function and 2.4.1b for a correspondence, as well as Figure 2.4.2a and Figure 2.4.2b for another interpretation which emphasizes “mapping”.

Example 50 A sequence is a function $f : \mathbb{N} \rightarrow B$ for some set B .

Definition 51 Let f be an arbitrary function with domain A and $R(f) \subset B$. If $E \subset A$, then the **(direct) image** of E under f , denoted $f(E)$, is the subset $\{f(a) | a \in E \cap D(f)\} \subset R(f)$. See Figure 2.4.3a.

Theorem 52 Let f be a function with domain A and $R(f) \subset B$ and let $E, F \subset A$. (a) If $E \subset F$, then $f(E) \subset f(F)$. (b) $f(E \cap F) \subset f(E) \cap f(F)$, (c) $f(E \cup F) = f(E) \cup f(F)$, (d) $f(E \setminus F) \subset f(E)$.

Proof. (a) If $a \in E$, then $a \in F$ so $f(a) \in f(F)$. But this is true $\forall a \in E$, hence $f(E) \subset f(F)$. ■

Exercise 2.4.1 Finish the proof of Theorem 52.

Definition 53 If $H \subset B$, then the **inverse image** of H under f , denoted $f^{-1}(H)$, is the subset $\{a | f(a) \in H\} \subset D(f)$. See Figure 2.4.3b.

It is important to note that the inverse image is different from the inverse function (to be discussed shortly). The inverse function need not exist when the inverse image does. See Example 65.

Theorem 54 Let $G, H \subset B$. (a) If $G \subset H$, then $f^{-1}(G) \subset f^{-1}(H)$. (b) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$, (c) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$, (d) $f^{-1}(G \setminus H) = f^{-1}(G) \setminus f^{-1}(H)$.

Proof. (a) If $a \in f^{-1}(G)$, then $f(a) \in G \subseteq H$ so $a \in f^{-1}(H)$. ■

Exercise 2.4.2 Finish the proof of Theorem 54.

Exercise 2.4.3 Let $f : A \rightarrow B$ be a function. Prove that the inverse images $f^{-1}(\{a\})$ and $f^{-1}(\{a'\})$ are disjoint. FIX

2.4.1 Restrictions and extensions

Example 55 Let $A = \mathbb{R} \setminus \{0\}$ and $f(a) = \frac{1}{a}$. Then $R(f) = \mathbb{R} \setminus \{0\}$. See Figure 2.4.4a

To deal with the above “hole” in the domain of f in Example 55, we can employ the idea of restricting or extending the function to a given set.

Definition 56 Let $\Delta \subset D(f)$. The **restriction of f to the set Δ** , which we will denote $f|_r\Delta$, is given by $\{(a, b) \in f : a \in \Delta\}$. Let $\Delta \supset D(f)$. The **extension of f on the set Δ** , which we will denote $f|_e\Delta$, is given by $\left\{ (a, b) : b = \begin{cases} f(a) & a \in D(f) \\ g(a) & a \in \Delta \setminus D(f) \end{cases} \right\}$.

Example 57 See Figure 2.4.4b for a restriction of $\frac{1}{a}$ to $\Delta = \mathbb{R}_{++}$ and Figure 2.4.4c for an extension of $\frac{1}{a}$ on $\Delta = \mathbb{R}$ is $\left\{ b = \begin{cases} \frac{1}{a} & a \neq 0 \\ 0 & a = 0 \end{cases} \right\}$.

Note that extensions are not generally unique.

2.4.2 Composition of functions

Definition 58 Let $f : A \rightarrow B$ and $g : B' \rightarrow C$. Let $R(f) \subset B'$. The **composition** $g \circ f$ is the function from A to C given by $g \circ f = \{(a, c) \in A \times C : \exists b \in R(f) \subset B' \ni (a, b) \in f \text{ and } (b, c) \in g\}$.⁸ See Figure 2.4.5.

Note that order matters, as the next example shows.

Example 59 Let $A \subset \mathbb{R}$, $f(a) = 2a$, and $g(a) = 3a^2 - 1$. Then $g \circ f = 3(2a)^2 - 1 = 12a^2 - 1$ while $f \circ g = 2(3a^2 - 1) = 6a^2 - 2$.

⁸Alternatively, we create a new function $h(a) = g(f(a))$.

2.4.3 Injections and inverses

Definition 60 $f : A \rightarrow B$ is **one-to-one** or an **injection** if whenever $(a, b) \in f$ and $(a', b) \in f$ for $a, a' \in D(f)$, then $a = a'$.⁹

Definition 61 Let f be an injection. If $g = \{(b, a) \in B \times A : (a, b) \in f\}$, then g is an injection with $D(g) = R(f)$ and $R(g) = D(f)$. The function g is called the **inverse** to f and denoted f^{-1} . See Figure 2.4.6.

2.4.4 Surjections and bijections

Definition 62 If $R(f) = B$, f maps A **onto** B (in this case, we call f a **surjection**). See Figure 2.4.7

Definition 63 $f : A \rightarrow B$ is a **bijection** if it is one-to-one and onto (or an injection and a surjection).

Example 64 Let $E = [0, 1] \subset A = \mathbb{R}$, $H = [0, 1] \subset B = \mathbb{R}$, and $f(a) = 2a$. See Figure 2.4.8. $R(f) = \mathbb{R}$ so that f is a surjection, the image set is $f(E) = [0, 2]$, the inverse image set is $f^{-1}(H) = [0, \frac{1}{2}]$, f is an injection and has inverse $f^{-1}(b) = \frac{1}{2}b$, and as a consequence of being one-to-one and onto, is a bijection. Notice that if $F = [-1, 0]$, then $f(F) \cap f(E) = \{0\}$ and $f(E \cap F) = f(0) = \{0\}$, so that in the special case of injections statement (b) of Theorem 52 holds with equality.

Example 65 Let $E = [0, 1] \subset A = \mathbb{R}$, $H = [0, 1] \subset B = \mathbb{R}$, and $f(a) = a^2$. See Figure 2.4.9. $R(f) = \mathbb{R}_+$ so that f is not a surjection, the image set is $f(E) = [0, 1]$, the inverse image set is $f^{-1}(H) = [-1, 1]$, f is not an injection (since, for instance, $f(-1) = f(1) = 1$), and is obviously not a bijection. However, the restriction of f to \mathbb{R}_+ or \mathbb{R}_- (in particular, let $f_+ \equiv f|_{\mathbb{R}_+}$ and $f_- \equiv f|_{\mathbb{R}_-}$) is an injection and $f_+^{-1}(b) = \sqrt{b}$ while $f_-^{-1}(b) = -\sqrt{b}$. Finally, notice that if $F = [-1, 0]$, then $f(F) \cap f(E) = [0, 1]$ but that $f(E \cap F) = f(0) = 0$, which is why we cannot generally prove equality in statement (b) of Theorem 52.

The next theorem shows that composition preserves surjection. It is useful to prove that statements about infinite sets.

⁹Alternatively, we can say f is one-to-one if $f(a) = f(a')$ only when $a = a'$.

Theorem 66 *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be surjections. Their composition $g \circ f$ is a surjection.*

Exercise 2.4.4 *Prove Theorem 66. Answer: We must show that for $g \circ f : A \rightarrow C$ given by $(g \circ f)(a) = g(f(a))$, it is the case that $\forall c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$. To see this, let $c \in C$. Since g is a surjection, $\exists b \in B$ such that $g(b) = c$. Similarly, since f is a surjection, $\exists a \in A$ such that $f(a) = b$. Then $(g \circ f)(a) = g(f(a)) = g(b) = c$.*

2.5 Finite and Infinite Sets

The purpose of this section is to compare sizes of sets with respect to the number of elements they contain. Take two sets $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. The number of elements of the set A (also called the *cardinality* of A , denoted $\text{card}(A)$) is three and of the set B is four. In this case we say that the set B is bigger than the set A .

It is hard, however, to apply this same concept in comparing, for instance, the set of all natural numbers \mathbb{N} with the set of all integers \mathbb{Z} . Both are infinite. Is the "infinity" that represents $\text{card}(\mathbb{N})$ smaller than the "infinity" that represents $\text{card}(\mathbb{Z})$? One might think the statement was true because there are integers that are not real numbers (e.g. $-1, -2, -3, \dots$). We will show however that this statement is false, but first we have to introduce a different concept of the size of a set known as countability and uncountability. To illustrate it, one of the authors placed a set of 3 coins in front of his 3 year old daughter and asked her "Is that collection of coins countable?". She proceeded to pick up the first coin with her right hand, put it in her left hand, and said '1', pick up the second coin, put it in her left hand, and said '2', and pick up the final coin, put it in her left hand, and said '3'. Thus, she put the set of coins into a one-to-one assignment with the first three natural numbers. We will now make use of one-to-one assignments between elements of two sets.

Definition 67 *Two sets A and B are equivalent if there is a bijection $f : A \rightarrow B$.*

Definition 68 *An initial segment (or section) of \mathbb{N} is the set $\Theta_n = \{i \in \mathbb{N} : i \leq n\}$.*

Definition 69 A set A is **finite** if it is empty or there exists a bijection $f : A \rightarrow \Theta_n$ for some $n \in \mathbb{N}$. In the former case A has zero elements and in the latter case A has n elements.

Lemma 70 Let B be a proper subset of a finite set A . There does not exist a bijection $f : A \rightarrow B$.

Proof. (Sketch) Since A is finite, $\exists f : A \rightarrow \Theta_n$. If B is a proper subset of A , then it contains $m < n$ elements. But there cannot be a bijection between n and m elements. ■

Exercise 2.5.1 Prove lemma 70 more formally. See lemma 6.1 in Munkres.

Lemma 70 says that a proper subset of a finite set cannot be equivalent with the whole set. This is quite clear. But is it true for any set? Let's consider $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and a proper subset $\mathbb{N} \setminus \{1\} = \{2, 3, 4, \dots\}$. We can construct a one-to-one assignment from \mathbb{N} onto $\mathbb{N} \setminus \{1\}$ (i.e. $1 \rightarrow 2, 2 \rightarrow 3, \dots$). Thus, in this case, it is possible for a set to be equivalent with its proper subset. Given Lemma 70, we must conclude the following.

Theorem 71 \mathbb{N} is not finite.

Proof. By contradiction. Suppose \mathbb{N} is finite. Then $f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{1\}$ defined by $f(n) = n + 1$ is a bijection of \mathbb{N} with a proper subset of itself. This contradicts Lemma 70. ■

Definition 72 A set A is **infinite** if it is not finite. It is **countably infinite** if there exists a bijection $f : \mathbb{N} \rightarrow A$.

Thus, \mathbb{N} is countably infinite since f can be taken to be the identity function (which is a bijection).

Definition 73 A set is **countable** if it is finite or countably infinite. A set that is not countable is **uncountable**.

Next we examine whether the set of integers, \mathbb{Z} , is countable. That is, are \mathbb{N} and \mathbb{Z} equivalent? This isn't apparent since $\mathbb{N} = \{1, 2, \dots\}$ has one end of the set that goes to infinity, while $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ has two ends of the set that go to infinity. But it is possible to reorganize \mathbb{Z} in a way that looks like \mathbb{N} since we can simply construct $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$. One can think of this set as being constructed from two rows $\{0, 1, 2, \dots\}$ and $\{-1, -2, \dots\}$ by alternating between the first and second rows. This is formalized in the next example.

Example 74 The set of integers, \mathbb{Z} , is countably infinite. The function $f : \mathbb{Z} \rightarrow \mathbb{N}$ defined by

$$f(z) = \begin{cases} 2z & \text{if } z > 0 \\ -2z + 1 & \text{if } z \leq 0 \end{cases}$$

is a bijection.

Exercise 2.5.2 Prove that a finite union of countable sets is countable.

Next we examine whether $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$. As in the preceding example where we had two rows, we can think about enumerating the set $\mathbb{N} \times \mathbb{N}$ in Figure 2.5.1. As in the preceding example, each row has infinitely many elements but now there are an infinite number of rows. Yet all of the elements of this "infinite matrix" can be enumerated if we start from $(1, 1)$ and then continue by following the arrows. This enumeration provides us with the desired bijection as shown next.

Example 75 The cartesian product $\mathbb{N} \times \mathbb{N}$ is countably infinite. First, let the bijection $g : \mathbb{N} \times \mathbb{N} \rightarrow A$, where $A \subset \mathbb{N} \times \mathbb{N}$ consists of pairs (x, y) for which $y \leq x$, be given by $g(x, y) = (x + y - 1, y)$. Next construct a bijection $h : A \rightarrow \mathbb{N}$ given by $h(x, y) = \frac{1}{2}(x - 1)x + y$. Then the composition $f = h \circ g$ is the desired bijection.

We can actually weaken the condition for proving countability of a given set A . The next theorem accomplishes this.

Theorem 76 Let A be a non-empty set. The following statements are equivalent: (i) There is a surjection $f : \mathbb{N} \rightarrow A$. (ii) There is an injection $g : A \rightarrow \mathbb{N}$. (iii) A is countable.

Proof. (Sketch) (i) \Rightarrow (ii). Given f , define $g : A \rightarrow \mathbb{N}$ by $g(a) =$ smallest element of $f^{-1}(\{a\})$. Since f is a surjection, the inverse image $f^{-1}(\{a\})$ is non-empty so that g is well defined. g is an injection since if $a \neq a'$, the sets $f^{-1}(\{a\})$ and $f^{-1}(\{a'\})$ are disjoint (recall Exercise 2.4.3), so their smallest elements are distinct proving $g : A \rightarrow \mathbb{N}$ is an injection.

(ii) \Rightarrow (iii). Since $g : A \rightarrow R(g)$ is a surjection by definition, $g : A \rightarrow R(g)$ is a bijection. Since $R(g) \subset \mathbb{N}$, A must be countable.

(iii) \Rightarrow (i). By definition. ■

Exercise 2.5.3 Finish parts (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) of the proof of Theorem 76. See Munkres 7.1 (USES WELL ORDERING).

Example 77 The set of positive rationals, \mathbb{Q}_{++} , is countably infinite. Define a surjection $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}_{++}$ by $g(n, m) = \frac{m}{n}$. Since $\mathbb{N} \times \mathbb{N}$ is countable (Example 75), there is a surjection $h : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$. Then $f = g \circ h : \mathbb{N} \rightarrow \mathbb{Q}_{++}$ is a surjection (Theorem 66) so by Theorem 76, \mathbb{Q}_{++} is countable.

The intuition for the preceding example follows simply from Figure 2.5.1 if you replace the “,” with “/”. That is, replace (1, 1) with the rational $\frac{1}{1}$, (1, 2) with the rational $\frac{1}{2}$, (3, 2) with the rational $\frac{3}{2}$, etc.

Theorem 78 A countable union of countable sets is countable.

Proof. Let $\{A_i, i \in \Lambda\}$ be an indexed family of countable sets where Λ is countable. Because each A_i is countable, for each i we can choose a surjection $f_i : \mathbb{N} \rightarrow A_i$. Similarly, we can choose a surjection $g : \mathbb{N} \rightarrow \Lambda$. Define $h : \mathbb{N} \times \mathbb{N} \rightarrow \cup_{i \in \Lambda} A_i$ by $h(n, m) = f_{g(n)}(m)$, which is a surjection. Since $\mathbb{N} \times \mathbb{N}$ is in bijective correspondence with \mathbb{N} (recall Example 75), the countability of the union follows from Theorem 76. ■

The next theorem provides an alternative proof of example 75.

Theorem 79 A finite product of countable sets is countable.

Proof. Let A and B be two non-empty, countable sets. Choose surjective functions $g : \mathbb{N} \rightarrow A$ and $h : \mathbb{N} \rightarrow B$. Then the function $f : \mathbb{N} \times \mathbb{N} \rightarrow A \times B$ defined by $f(n, m) = (g(n), h(m))$ is surjective. By Theorem 76, $A \times B$ is countable. Proceed by induction for any finite product. ■

While it's tempting to think that this result could be extended to show that a countable product of countable sets is countable, the next Theorem shows this is false. Furthermore, it gives us our first example of an uncountable set.

Theorem 80 Let $X = \{0, 1\}$. The set of all functions $\mathbf{x} : \mathbb{N} \rightarrow X$, denoted X^ω , is uncountable.¹⁰

¹⁰An alternative statement of the theorem is that the set of all infinite sequences of X is uncountable.

Proof. We show that any function $g : \mathbb{N} \rightarrow X^\omega$ is not a surjection. Let $g(n) = (x_{n1}, x_{n2}, \dots, x_{nm}, \dots)$ where each x_{ij} is either 0 or 1. Define a point $\mathbf{y} = (y_1, y_2, \dots, y_n, \dots)$ of X^ω by letting

$$y_n = \begin{cases} 0 & \text{if } x_{nn} = 1 \\ 1 & \text{if } x_{nn} = 0 \end{cases}.$$

Now $\mathbf{y} \in X^\omega$ and \mathbf{y} is not in the image of g . That is, given n , $g(n)$ and \mathbf{y} differ in at least one coordinate, namely the n^{th} . Thus g is not a surjection. ■

The diagonal argument used above (See Figure 2.5.2) will be useful to establish the uncountability of the reals, which we save until Chapter 3.

Exercise 2.5.4 Consider the following game known as “matching pennies”. You (A) and I (B) each hold a penny. We simultaneously reveal either “heads” (H) or “tails” (T) to each other. If both faces match (i.e. both heads or both tails) you receive a penny, otherwise I get the penny. The action sets for each player are $S_A = S_B = \{H, T\}$. Now suppose we decide to play this game every day for the indefinite (infinite) future (we’re optimistic about medical technology). Before you begin, you should think of all the different combinations of actions you may employ in the infinitely repeated game. For instance, you may alternate H and T starting with H in the first round. Prove that although the number of actions you play in the infinitely repeated game is countable and the set of actions S_A is finite, the set of possible combinations of actions ($S_A \times S_A \times \dots$) is uncountable.

2.6 Algebras of Sets

An algebra is just a collection of sets (which could be infinite) that is closed under (finite) union and complementation. It is used extensively in probability and measure theory.

Definition 81 A collection \mathcal{A} of subsets of X is called an **algebra** of sets if (i) $A^c \in \mathcal{A}$ if $A \in \mathcal{A}$ and (ii) $A \cup B \in \mathcal{A}$ if $A, B \in \mathcal{A}$.

Note that $\emptyset, X \in \mathcal{A}$ since, for instance, $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ by (i) and then $A \cup A^c = X \in \mathcal{A}$ by (ii). It also follows from De Morgan’s laws that (iii) $A \cap B \in \mathcal{A}$ if $A, B \in \mathcal{A}$. The definition extends to larger collections (just take unions two at a time).

Theorem 82 *Given any collection \mathcal{C} of subsets of X , there is a smallest algebra \mathcal{A} which contains \mathcal{C} .*

Proof. (Sketch) It is sufficient to show there is an algebra \mathcal{A} containing \mathcal{C} such that if \mathcal{B} is any algebra containing \mathcal{C} , then $\mathcal{B} \supset \mathcal{A}$. Let \mathcal{F} be the family of all algebras that contain \mathcal{C} (which is nonempty since $\mathcal{P}(X) \in \mathcal{F}$). Let $\mathcal{A} = \cap\{\mathcal{B} : \mathcal{B} \in \mathcal{F}\}$. Then \mathcal{C} is a subcollection of \mathcal{A} since each \mathcal{B} in \mathcal{F} contains \mathcal{C} . All that remains to be shown is that \mathcal{A} is an algebra (i.e. if A and B are in \mathcal{A} , then $A \cup B$ and A^c are in $\cap\{\mathcal{B} : \mathcal{B} \in \mathcal{F}\}$). It follows from the definition of \mathcal{A} that $\mathcal{B} \supset \mathcal{A}$. See Figure 2.6.1 ■

Exercise 2.6.1 *Finish the proof of Theorem 82. If A and B are in \mathcal{A} , then for each $\mathcal{B} \in \mathcal{F}$, we have $A \in \mathcal{B}$ and $B \in \mathcal{B}$. Since \mathcal{B} is an algebra, $A \cup B \in \mathcal{B}$. Since this is true for every $\mathcal{B} \in \mathcal{F}$, we have $A \cup B \in \cap\{\mathcal{B} | \mathcal{B} \in \mathcal{F}\}$. Similarly, if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.*

We say that the smallest algebra containing \mathcal{C} is called *the algebra generated by \mathcal{C}* . By construction, the smallest algebra is unique. Notice the proof makes clear that the intersection of any collection of algebras is itself an algebra.

Example 83 *Let $X = \{a, b, c\}$. The following three collections are algebras: $\mathcal{C}_1 = \{\emptyset, X\}$, $\mathcal{C}_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\mathcal{C}_3 = \mathcal{P}(X)$. The following two collections are not algebras: $\mathcal{C}_4 = \{\emptyset, \{a\}, X\}$ since, for instance, $\{a\}^c = \{b, c\} \notin \mathcal{C}_4$ and $\mathcal{C}_5 = \{\emptyset, \{a\}, \{b\}, \{b, c\}, X\}$ since $\{b\}^c = \{a, c\} \notin \mathcal{C}_5$. However, the smallest algebra which contains \mathcal{C}_4 is just \mathcal{C}_2 . To see this, we can apply the argument in Theorem 82. Let $\mathcal{F} = \{\mathcal{C}_2, \mathcal{P}(X)\}$ be the family of all algebras that contain \mathcal{C}_4 . But $\mathcal{A} = \mathcal{C}_2 \cap \mathcal{P}(X) = \mathcal{C}_2$.*

Exercise 2.6.2 *Let $X = \mathbb{N}$. Show that the collection $\mathcal{A} = \{A_i : A_i \text{ is finite or } \mathbb{N} \setminus A_i \text{ is finite}\}$ is an algebra on \mathbb{N} and that it is a proper subset of $\mathcal{P}(\mathbb{N})$.*

The next theorem proves that it is always possible to construct a new collection of disjoint sets from an existing algebra with the property that its union is equivalent to the union of subsets in the existing algebra. This will become very useful when we begin to think about probability measures.

Theorem 84 *Let \mathcal{A} be an algebra comprised of subsets $\{A_i : i \in \Lambda\}$.¹¹ Then there is a collection of subsets $\{B_i : i \in \Lambda\}$ in \mathcal{A} such that $B_n \cap B_m = \emptyset$ for $n \neq m$ and $\cup_{i \in \Lambda} B_i = \cup_{i \in \Lambda} A_i$.*

¹¹Note that the index set Λ can be countably or even uncountably infinite.

Proof. (Sketch) The theorem is trivial when the collection is finite (see Example 85 below). When the collection is indexed on \mathbb{N} , we let $B_1 = A_1$ and for each $n \in \mathbb{N} \setminus \{1\}$ define

$$\begin{aligned} B_n &= A_n \setminus [A_1 \cup A_2 \cup \dots \cup A_{n-1}] \\ &= A_n \cap A_1^c \cap A_2^c \cap \dots \cap A_{n-1}^c. \end{aligned}$$

Since the complements and intersections of sets in \mathcal{A} are in \mathcal{A} , $B_n \in \mathcal{A}$ and by construction $B_n \subset A_n$. The remainder of the proof amounts to showing that the above constructed sets are disjoint and yield the same union as the algebra. ■

Exercise 2.6.3 *Finish the proof of Theorem 84 above. See Royden Prop 2 p. 17.*

Note that Theorem 84 does not say that the new collection $\{B_i : i \in \Lambda\}$ is necessarily itself an algebra. The next example shows this.

Example 85 *Let $X = \{a, b, c\}$ and algebra $\mathcal{A} = \mathcal{P}(X)$ with $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{c\}$, $A_4 = \{a, b\}$, $A_5 = \{a, c\}$, $A_6 = \{b, c\}$, $A_7 = \emptyset$, $A_8 = X$. Let $B_1 = A_1$. By construction $B_2 = A_2 \setminus A_1 = \{b\}$, $B_3 = A_3 \setminus \{A_1 \cup A_2\} = \{c\}$, $B_n = A_n \setminus \{A_1 \cup A_2 \cup \dots \cup A_{n-1}\} = \emptyset$ for $n \geq 4$. Note that the new collection $\{\{a\}, \{b\}, \{c\}\}$ is not itself an algebra, since it's not closed under complementation and that if we chose a different sequence of A_i we could obtain a different collection $\{B_i : i \in \Lambda\}$.*

In the next chapter, we will learn an important result: any (open) set of real numbers can be represented as a countable union of disjoint open intervals. Hence we cannot guarantee that the set is in an algebra, which is closed only under finite union, even if all the sets belong to the algebra. Thus we extend the notion of an algebra to countable collections that are closed under complementation and countable union.

Definition 86 *A collection \mathcal{X} of subsets of X is called a σ -algebra of sets if (i) $A^c (= X \setminus A) \in \mathcal{X}$ if $A \in \mathcal{X}$ and (ii) $\cup_{n \in \mathbb{N}} A_n \in \mathcal{X}$ if each $A_n \in \mathcal{X}$.*

As in the case of algebras, $\emptyset, X \in \mathcal{X}$ and $\cap_{n=1}^{\infty} A_n = (\cup_{n=1}^{\infty} A_n^c)^c \in \mathcal{X}$ which means that a σ -algebra is closed under countable intersections as well.

Furthermore, we can always construct the unique smallest σ -algebra containing a given collection \mathcal{X} (called the σ -algebra generated by \mathcal{X}) by forming the intersection of all the σ -algebras containing \mathcal{X}). This result is an extension of Theorem 82.

Theorem 87 *Given any collection \mathcal{X} of subsets of X , there is a smallest σ -algebra that contains \mathcal{X} .*

Exercise 2.6.4 *Prove Theorem 87.*

Exercise 2.6.5 *Let \mathcal{C} , \mathcal{D} be collections of subsets of X . (i) Show that the smallest algebra generated by \mathcal{C} is contained in the smallest σ -algebra generated by \mathcal{C} . (ii) If $\mathcal{C} \subset \mathcal{D}$, show the smallest σ -algebra generated by \mathcal{C} is contained in the smallest σ -algebra generated by \mathcal{D} .*

Figures for Sections 2.1 to 2.2

Figure 2.1.1: Set Operations

Figure 2.1.2: Distributive Property

Figure 2.1.3: DeMorgan's Laws

Figure 2.2.1: Cartesian Product

Figures for Sections 2.3 to 2.4

Figure 2.3.1: Illustrating Partial vs Total Ordering

Figure 2.3.2: A Lattice

Figure 2.3.3: A Partially Ordered Set that's not a Lattice

Figure 2.3.4: Chains and Upper Bounds

Figure 2.4.1a: $f : A \rightarrow B$ and 2.4.1b: Not a functionFigure 2.4.2a: Graph of f in $A \times B$ and 2.4.2b: Not a function in $A \times B$

Figure 2.4.3a: Image Set and Figure 2.4.3b: Inverse Image Set

Figure 2.4.4a: Graph of $\frac{1}{a}$ Figure 2.4.4b: Restriction of $\frac{1}{a}$ and Figure 2.4.4c: Extension of $\frac{1}{a}$

Figure 2.4.5: Composition

Figure 2.4.6: Inverse Function

Figure 2.4.7. Surjection

Figure 2.4.8. Graph of $2a$ Figure 2.4.9. Graph of a^2 **Figures for Section 2.5**Figure 2.5.1: Countability of $\mathbb{N} \times \mathbb{N}$ Figure 2.5.2: Uncountability of $\{0, 1\}^\omega$ **Figures for Section 2.6**Figure 2.5.1: Smallest σ -algebras

2.7 Bibliography for Chapter 2

Sections 2.1 to 2.2 drew on Bartle (1978, Ch 1), Munkres (1975, Ch.1), and Royden (1988, Ch 1, Sec. 1,3,4). The material on relations and correspondences in Section 2.3 is drawn from Royden (1988, Ch1, Sec 7), Munkres (1975, Ch1, Sec 3), Aliprantis and Border (1999, Ch.1, Sec 2), and Mas-Collel, Whinston, and Green (1995, Ch 1, Sec B). The material on functions in Section 5.2 is drawn from Bartle (1976, Ch 2) and Munkres (1975, Ch1, Sec 2). Section 2.5 drew from Munkres (1975, Ch1, Sec 6-7) and Bartle (1976, Ch.3).

2.8 End of Chapter Problems.

1. Let $f : A \rightarrow B$ be a function. Prove the following statements are equivalent.
 - (i) f is one-to-one on A .
 - (ii) $f(C \cap D) = f(C) \cap f(D)$ for all subsets C and D of A .
 - (iii) $f^{-1}[f(C)] = C$ for every subset C of A .
 - (iv) For all disjoint subsets C and D of A , the images $f(C)$ and $f(D)$ are disjoint.
 - (v) For all subsets C and D of A with $D \subseteq C$, we have $f(C \setminus D) = f(C) \setminus f(D)$.
2. Prove that a finite union of countable sets is a countable set.

Chapter 3

The Space of Real Numbers

In this chapter we introduce the most common set that economists will encounter. The real numbers can be thought of as being built up using the set operations and order relations that we introduced in the preceding chapter. In particular we can start with the most elementary set \mathbb{N} (the counting numbers we all learned in pre-kindergarten) upon which certain operations like '+' and '.' are defined. The naturals are closed (i.e. for any two counting numbers, say n_1 and n_2 , the operation $n_1 + n_2$ is contained in \mathbb{N}). However, \mathbb{N} is not closed with respect to certain other operations like '-' since for example $2 - 4 \notin \mathbb{N}$. To handle that example we need the integers \mathbb{Z} , which is closed under '+', '.', and '-' (i.e. $2 - 4 \in \mathbb{Z}$). However, \mathbb{Z} can't handle operations like dividing 2 pies between 3 people (i.e. $\frac{2}{3} \notin \mathbb{Z}$). To handle that example we need the rationals \mathbb{Q} , which is closed under '+', '-', '.', and '÷'. (i.e. $\frac{2}{3} \in \mathbb{Q}$). But the rationals can't handle something as simple as finding the length of the diagonal of a unit square. That is, $\sqrt{2} \notin \mathbb{Q}$. To extend \mathbb{Q} to include such cases, besides the operations '+', '-', '.', and '÷', we could use Dedekind cuts which makes use the order relation ' \leq '. A Dedekind cut in \mathbb{Q} is an ordered pair (D, E) of nonempty subsets of \mathbb{Q} with the properties $D \cap E = \emptyset$, $D \cup E$, and $d < e$, $\forall d \in D$ and $\forall e \in E$. An example of a cut in \mathbb{Q} is, for $\xi \in \mathbb{Q}$,

$$D = \{x \in \mathbb{Q} : x \leq \xi\}, E = \{x \in \mathbb{Q} : x > \xi\}.$$

In this case, we say that $\xi \in \mathbb{Q}$ represents the cut (D, E) . If a cut can be represented by a rational number, it is called a rational cut. It is simple to see that there are cuts in \mathbb{Q} which cannot be represented by a rational

number. For example, take the cut

$$D' = \{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 \leq 2\}, \quad E' = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}.$$

As we will show in this chapter, (D', E') cannot be represented by a rational number. Such cuts are called irrational cuts. Each irrational cut defines a unique number. The set of all such numbers is called the irrational numbers. In this way, we can extend the rationals by adding in these irrationals.

Rather than build up the real numbers as discussed above, our approach will simply be to take the real numbers as given, list a set of axioms for them, and derive properties of the real numbers as consequences of these axioms. The first group of axioms describe the algebraic properties, the second group the order properties, and we shall call the third the completeness axiom. With these three groups of axioms we can completely characterize the real numbers. In the next chapter we will focus on important issues like convergence, compactness, completeness, and connectedness in spaces more general than the real numbers. However, to understand those concepts it is often helpful to provide examples from \mathbb{R} , which is why we start here.

In this chapter we focus on four important results in \mathbb{R} . The first (see Theorem 108) is that any open set in \mathbb{R} can be written in terms of a countable union of open intervals. The next two results are proven using the Nested Intervals Property (see Theorem 116) which says that a decreasing sequence of closed, bounded, nonempty intervals “converges” to a nonempty set. The first important result that this is used to prove is the Bolzano-Weierstrass Theorem (118) which says that every bounded infinite subset of \mathbb{R} has a point with sufficiently many points in any subset around it. It is also used to prove the important “size” result (see Theorem 122) that open intervals in \mathbb{R} are uncountable.

3.1 The Field Axioms

The functions or binary operations “+” and “·” on $\mathbb{R} \times \mathbb{R}$ to \mathbb{R} satisfy the following axioms. It shouldn’t be surprising that we require the operations to satisfy commutative, associative, and distributive properties as we did in Chapter 2 with respect to the set operations ‘ \cup ’ and ‘ \cap ’.

Axiom 1 (Algebraic Properties of \mathbb{R}) $x, y, z \in \mathbb{R}$ satisfy:

A1. $x + y = y + x.$

A2. $(x + y) + z = x + (y + z)$

A3. $\exists 0 \in \mathbb{R} \ni x + 0 = x, \forall x \in \mathbb{R}$

A4. $\forall x \in \mathbb{R}, \exists w \in \mathbb{R} \ni x + w = 0$

A5. $x \cdot y = y \cdot x$

A6. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

A7. $\exists 1 \in \mathbb{R} \ni 1 \neq 0$ and $x \cdot 1 = x, \forall x \in \mathbb{R}$

A8. $\forall x \in \mathbb{R} \ni x \neq 0, \exists w \in \mathbb{R} \ni x \cdot w = 1$

A9. $x \cdot (y + z) = x \cdot y + x \cdot z$

Any set that satisfies Axiom 1 is called a **field** (under “+” and “·”). If we have a field, we can perform all the operations of elementary algebra, including the solution of simultaneous linear equations. It follows from A1 that the 0 in A3 is unique, which was used in formulating A4, A7, and A8. It also follows that the w in A4 is unique and denoted “ $-x$ ”. Subtraction “ $x - y$ ” is defined as “ $x + (-y)$ ”. That 1 in A7 is unique follows from A5. The w in A8 can also be shown to be unique.

Exercise 3.1.1 *Let $a, b \in \mathbb{R}$. Prove that the equation $a + x = b$ has the unique solution $x = (-a) + b$. With $a \neq 0$, prove that the equation $a \cdot x = b$ has the unique solution $x = \left(\frac{1}{a}\right) \cdot b$. (This is Theorem 4.4 of Bartle).*

In what follows, we drop the “·” to denote multiplication and write xy for $x \cdot y$. Furthermore, we write x^2 for xx and generally $x^{n+1} = (x^n)x$ with $n \in \mathbb{N}$. It follows by mathematical induction that $x^{n+m} = x^n x^m$ for $x \in \mathbb{R}$ and $n, m \in \mathbb{N}$. We shall also write $\frac{x}{y}$ instead of $\left(\frac{1}{y}\right) \cdot x$. Recall that we defined the rationals as $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\right\}$.

Theorem 88 *There does not exist a rational number $q \in \mathbb{Q}$ such that $q^2 = 2$.*

Proof. Suppose not. Then $\left(\frac{m}{n}\right)^2 = 2$ for $m, n \in \mathbb{Z}$, $n \neq 0$. Assume, without loss of generality, that m and n have no common factors. Since $m^2 = 2n^2$ is an even integer, then m must be an even integer.¹ In that case we can represent it as $m = 2k$ for some integer k . Hence $(m^2 =) 4k^2 = 2n^2$ or $n^2 = 2k^2$ which implies that n is also even. But this implies that m and n are both divisible by 2, which contradicts the assumption that m and n have no common factors. ■

Theorem 88 says that the cut (D', E') in the introduction to this chapter is not rational.

Definition 89 *All of the elements of \mathbb{R} which are not rational numbers are **irrational** numbers.*

In Section 3.3 we provide a complementary result to Theorem 88 to establish the existence of irrational numbers.

3.2 The Order Axioms

The next class of properties possessed by the real numbers have to do with the fact that they are ordered. The order relation ' \leq ' defined on \mathbb{R} is a special, and most important, case of the more general relations discussed in Chapter 2.²

Axiom 2 (Order Properties of \mathbb{R}) *The subset P of positive real numbers satisfies³*

B1. If $x, y \in P$, then $x + y \in P$.

B2. If $x, y \in P$, then $x \cdot y \in P$.

B3. If $x \in \mathbb{R}$, then one and only one of the following holds: $x \in P$, $x = 0$, or $-x \in P$.

Note that B3 implies that if $x \in P$, then $-x \notin P$. More importantly, B3 guarantees that \mathbb{R} is totally ordered with respect to the order relation ' \leq '.

¹Otherwise, if m is odd we can represent it as $m = 2k + 1$ for some integer k . But then $m^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd, contradicting the fact that m^2 is even.

²In fact, the order relations in Chapter 2 were developed to generalize these concepts to more abstract spaces than \mathbb{R} .

³Later, we will associate P with the notation \mathbb{R}_{++} .

Definition 90 Any system satisfying Axiom 1 and Axiom 2 is called an *ordered field*.

By definition then, \mathbb{R} is an ordered field.

Definition 91 Let $x, y \in \mathbb{R}$. If $x - y \in P$, then we say $x > y$ and if $x - y \in P \cup \{0\}$, then we say $x \geq y$. If $-(x - y) \in P$, then we say $x < y$ and if $-(x - y) \in P \cup \{0\}$, then we say $x \leq y$.

Exercise 3.2.1 Show that (\mathbb{R}, \leq) is a totally ordered set.

The following properties are a consequence of Axiom 2.

Theorem 92 Let $x, y, z \in \mathbb{R}$. (i) If $x > y$ and $y > z$, then $x > z$. (ii) Exactly one holds: $x > y$, $x = y$, $x < y$. (iii) If $x \geq y$ and $y \geq x$, then $x = y$.

Proof. (i) If $x - y \in P$ and $y - z \in P$, then $B1 \Rightarrow (x - y) + (y - z) \in P$ or $(x - z) \in P$ ■

Exercise 3.2.2 Finish the proof of Theorem 92. (Bartle 5.4)

The next theorem is one of the simplest we will encounter, yet it is one of the most far-reaching. For one thing, it implies that given any strictly positive real number, there is another smaller and strictly positive real number so that there is no smallest strictly positive real number!⁴

Theorem 93 (Half the distance to the goal line) If $x, y \in \mathbb{R}$ with $x > y$, then $x > \frac{1}{2}(x + y) > y$.

Proof. $x > y \Rightarrow x + x > x + y$ and $x + y > y + y \Rightarrow 2x > x + y > 2y$. ■

Now we define a very useful function on \mathbb{R} that assigns to each real number its distance from the origin.

Definition 94 If $x \in \mathbb{R}$, the *absolute value* of x , denoted $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}_+$, is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

⁴Another thing it proves is that even if a defense is continuously penalized half the distance to the goal line, the offense will never score unless they finally run a play.

This function satisfies the well-known property of triangles; that is, the length of any side of a triangle is less than the sum of the lengths of the other two sides.

Theorem 95 (Triangle Inequality) *If $x, y \in \mathbb{R}$, then $|x + y| \leq |x| + |y|$.*

Proof. (Sketch) Since $x \leq |x|$ and $y \leq |y|$, then $|x| - x \in P \cup \{0\}$ and $|y| - y \in P \cup \{0\}$. By Axiom B1, $(|x| - x) + (|y| - y) \in P \cup \{0\}$. But $(|x| - x) + (|y| - y) = (|x| + |y|) - (x + y)$, so $(|x| + |y|) - (x + y) \in P \cup \{0\}$ or $(x + y) \leq (|x| + |y|)$. But then $|x + y| \leq |x| + |y|$. ■

3.3 The Completeness Axiom

This axiom distinguishes \mathbb{R} from other totally ordered fields like \mathbb{Q} . To begin, we use the definition of upper and lower bounds from 40 with the order relation ' \leq ' on \mathbb{R} . If $S \subset \mathbb{R}$ has an upper and/or lower bound, it has infinitely many (e.g. if u is an ub of S , then $u + n$ is an ub for $n \in \mathbb{N}$). Supremum and infimum that were defined in 40 for the general case, can be characterized in \mathbb{R} by the following lemma.

Lemma 96 *Let $S \subset \mathbb{R}$. Then $u \in \mathbb{R}$ is a **supremum** (or **sup** or **least upper bound** (lub)) of S iff (i) $s \leq u, \forall s \in S$ and (ii) $\forall \varepsilon > 0, \exists s \in S$ such that $u - \varepsilon < s$.⁵ Similarly, $\ell \in \mathbb{R}$ is an **infimum** (or **inf** or **greatest lower bound** (glb)) of S iff (i) $\ell \leq s, \forall s \in S$ and (ii) $\forall \varepsilon > 0, \exists s \in S$ such that $s < \ell + \varepsilon$. See Figure 3.3.1.*

Proof. (\Rightarrow) (i) holds by definition (just use ' \leq ' on \mathbb{R} in 40). To see (ii), suppose u is the least upper bound. Because $u - \varepsilon < u$, then $u - \varepsilon$ cannot be an upper bound. This implies $\exists s \in S$ such that $u - \varepsilon < s$.

(\Leftarrow) (i) implies u is an upper bound again by definition. To see that (ii) implies u is the least upper bound, consider $v < u$. Then $u - v = \varepsilon > 0$ (or $v = u - \varepsilon$). By (ii), $\exists s \in S$ such that $u - \varepsilon = v < s$, hence v is not an upper bound. ■

In the case where S does not have an upper (lower) bound, we assign $\sup S = \infty$ ($\inf S = -\infty$).

⁵Statement (i) makes u an ub while (ii) makes it the lub.

Example 97 *A set may not contain its sup. To see this, let $S = \{x \in \mathbb{R} : 0 < x < 1\}$ and $S' = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Any number $u \geq 1$ is an ub for both sets, but while S' contains the ub 1, S does not contain any of its ub! Also, it's clear that no number $c < 1$ can be an ub for S . To see this, just apply our famous Theorem 93. That is, since $c < 1$, then $\exists s = \frac{1+c}{2} > c$ and $s \in S$.*

Theorem 98 *There can be only one supremum for any $S \subset \mathbb{R}$.*

Proof. If u_1 and u_2 are lub, then they are both ub. Since u_1 is lub and u_2 is ub, then $u_1 \leq u_2$. Similarly, since u_2 is lub and u_1 is ub, then $u_2 \leq u_1$. Then $u_1 = u_2$. ■

The next axiom is critical to establish that \mathbb{R} does not have any “holes” in it. In particular, it will be sufficient to establish that the set \mathbb{R} is “complete” (a term that will be made precise in Chapter 4). Don't be fooled however, it takes more work than just stating the Axiom to establish completeness.

Axiom 3 (Completeness Property of \mathbb{R}) *Every non-empty set $S \subset \mathbb{R}$ which has an upper bound has a supremum.*

From the completeness axiom, it is easy to establish that every non-empty set which has a lower bound has an infimum.

A consequence of Axiom 3 is that \mathbb{N} (a subset of \mathbb{R}) is not bounded above in \mathbb{R} .

Theorem 99 (Archimedean Property) *If $x \in \mathbb{R}$, $\exists n_x \in \mathbb{N}$ such that $x < n_x$.*

Proof. Suppose not. Then x is an ub for \mathbb{N} and hence by Axiom 3 \mathbb{N} has a sup, call it u , and $u \leq x$.⁶ Since $u - 1$ is not an ub, $\exists n_1 \in \mathbb{N} \ni u - 1 < n_1$. Then $u < n_1 + 1$ and since $n_1 + 1 \in \mathbb{N}$, this contradicts that u is an ub of \mathbb{N} . ■

It follows from Theorem 99 that there exists a rational number between any two real numbers.

Theorem 100 *If $x, y \in \mathbb{R}$ with $x < y$, $\exists q \in \mathbb{Q}$ such that $x < q < y$.*

⁶Note that u is not necessarily in \mathbb{N} , which is why we choose to subtract $1 \in \mathbb{N}$ in the next statement.

Exercise 3.3.1 Prove Theorem 100. (Royden p.35)

The following theorem complements the result that there are elements of \mathbb{R} which are not rational in Theorem 88 of section 3.1. It provides an existence proof of an irrational. We present it since it makes use of Axiom 3. Without the Axiom, the set $S = \{y \in \mathbb{R}_+ : y^2 \leq 2\}$ does not have a supremum.

Theorem 101 $\exists x \in \mathbb{R}_+$ such that $x^2 = 2$.

Proof. Let $S = \{y \in \mathbb{R}_+ : y^2 \leq 2\}$. Clearly S is non-empty (take 1) and S is bounded above (take 1.5). Let $x = \sup S$, which exists by Axiom 3. Suppose $x^2 \neq 2$. Then either $x^2 < 2$ or $x^2 > 2$. First take $x^2 < 2$. Let $n \in \mathbb{N}$ be sufficiently large so that $\frac{2x+1}{n} < 2 - x^2$. Then $(x + \frac{1}{n})^2 \leq x^2 + \frac{2x+1}{n} < 2$.⁷ This means $x + \frac{1}{n} \in S$ which contradicts that x is an upper bound. Next take $x^2 > 2$. Let $m \in \mathbb{N}$ be sufficiently large so that $\frac{2x}{m} < x^2 - 2$. Then $(x - \frac{1}{m})^2 > x^2 - \frac{2x}{m} > 2$. Since $x = \sup S$, then $\exists s' \in S$ such that $x - \frac{1}{m} < s'$. But this implies $(s')^2 > (x - \frac{1}{m})^2$ (or $(s')^2 > 2$) which contradicts $s' \in S$. ■

Exercise 3.3.2 Why doesn't $S = \{y \in \mathbb{R}_+ : y^2 + 1 \leq 0\}$ work?

The next theorem complements the result in Theorem 100 and establishes that between any two real numbers there exists an irrational number.

Theorem 102 Let $x, y \in \mathbb{R}$ with $x < y$. If ι is any irrational number, then $\exists q \in \mathbb{Q}$ such that the irrational number ιq satisfies $x < \iota q < y$.

Exercise 3.3.3 Prove Theorem 102. (Bartle)

In fact, there are infinitely many of both kinds of numbers between x and y

⁷The first weak inequality holds with equality only if $n = 1$.

3.4 Open and Closed Sets

In this section we define the most common subsets of real numbers and determine some of their properties.

Definition 103 If $a, b \in \mathbb{R}$, then the set $\{x \in \mathbb{R} : a < x < b\}$ ($\{x \in \mathbb{R} : a \leq x \leq b\}$, $\{x \in \mathbb{R} : a \leq x < b\}$) is called a **open (closed, half-open) cell** denoted (a, b) ($[a, b], [a, b)$) respectively with endpoints a and b . If $a \in \mathbb{R}$, then the set $\{x \in \mathbb{R} : a < x\}$ ($\{x \in \mathbb{R} : a \leq x\}$) is called an **open (closed) ray** denoted (a, ∞) ($[a, \infty)$), respectively. An **interval** in \mathbb{R} is either a cell, a ray, or all of \mathbb{R} .

A generalization of the notion of an open interval is that of an open set.

Definition 104 A set $O \subset \mathbb{R}$ is **open** if for each $x \in O$, there is some $\delta > 0$ such that the open interval $B_\delta(x) = \{y \in O : |x - y| < \delta\} \subset O$.

Example 105 $(0, 1) \subset \mathbb{R}$ is open since for any x arbitrarily close to 1 (i.e. $x = 1 - \varepsilon$, $\varepsilon > 0$ arbitrarily small), there is an open interval $B_{\frac{\varepsilon}{2}}(1 - \varepsilon) \subset (0, 1)$ by Theorem 93. $(0, 1]$ is not open since there does not exist $\delta > 0$ for which $B_\delta(1) \subset (0, 1]$. That is, no matter how small $\delta > 0$ is, there exists $x' = 1 + \frac{\delta}{2} \in B_\delta(1)$ by Theorem 93 which is not contained in $(0, 1]$. See Figure 3.4.1.

Theorem 106 (i) \emptyset and \mathbb{R} are open. (ii) The intersection of any finite collection of open sets in \mathbb{R} is open. (iii) The union of any collection of open sets in \mathbb{R} is open.

Proof. (i) \emptyset contains no points, hence Definition 104 is trivially satisfied.⁸ \mathbb{R} is open since all $y \neq x$ are already in \mathbb{R} .

(ii) Let $\{O_i : O_i \subset \mathbb{R}, O_i \text{ open}, i = 1, \dots, k\}$ be a finite collection of open sets. We must show $O = \bigcap_{i=1}^k O_i$ is open. Assume $x \in \mathbb{R}$. By definition of an intersection, $x \in O_i, \forall i = 1, \dots, k$. Since each O_i is open, we can find $B_{\delta_i}(x) \subset O_i$ for each i . Let $\delta = \min\{\delta_i : i = 1, \dots, k\}$. Then $B_\delta(x) \subset B_{\delta_i}(x) \subset O_i, \forall i$. This implies $B_\delta(x) \subset O$.

(iii) Take $x \in O = \bigcup_{i \in \Lambda} O_i$, where Λ is either a finite or infinite index set. Since O_i is open, $\exists B_\delta(x) \subset O_i \subset \bigcup_{i \in \Lambda} O_i$. ■

⁸In particular, the statement $x \in \emptyset$ is always false. Thus, according to the truth table, any implication of the form $x \in \emptyset \Rightarrow P(x)$ is true.

Example 107 Property (ii) of Theorem 106 does not necessarily hold for infinite intersections. Consider the following counterexample. Let $O_n = \{x \in \mathbb{R} : -\frac{1}{n} < x < \frac{1}{n}, n \in \mathbb{N}\}$. Then $\bigcap_{n=1}^{\infty} O_n = \{0\}$, but a singleton set is not open since there does not exist $\delta > 0$ such that $B_\delta(0) \subset \{0\}$. See Figure 3.4.2.

The following theorem provides a characterization of open sets in \mathbb{R} .

Theorem 108 (Open Sets Property in \mathbb{R}) Every open set in \mathbb{R} is the union of a countable collection of disjoint open intervals.

Proof. The proof is in several steps. First, construct an open interval around each $y \in O$. Let O be open. Then, for each $y \in O$, \exists an open interval (x, z) such that $x < y < z$ and $(x, z) \subset O$. Let $b = \sup\{z : (y, z) \subset O\}$ and $a = \inf\{x : (x, y) \subset O\}$. Then $a < y < b$ and $I_y = (a, b)$ is an open interval containing y .

Second, show the constructed interval is contained in O . Take any $w \in (a, b)$ with $w > y$. Then $y < w < b$ and by the definition of b (i.e. it is the sup), we know $w \in O$. An identical argument establishes that if $w < y$, $w \in O$.

Third, show the constructed interval is open (i.e. $a, b \notin O$). If $b \in O$, then since O is open, $\exists \varepsilon > 0$ such that $(b - \varepsilon, b + \varepsilon) \subset O$ and hence $(y, b + \varepsilon) \subset O$ which contradicts the definition of b .

Fourth, show the union of constructed intervals is O . Let $w \in O$. Then $w \in I_w$ and hence $w \in \bigcup_{y \in O} I_y$.

Fifth, establish that the intervals are disjoint. Suppose $y \in (a_1, b_1) \cap (a_2, b_2)$. Since $b_1 = \sup\{z : (y, z) \subset O\}$ and $(y, b_2) \subset O$, then $b_1 \leq b_2$. Since $b_2 = \sup\{z : (y, z) \subset O\}$ and $(y, b_1) \subset O$, then $b_2 \leq b_1$. But $b_1 \leq b_2$ and $b_1 \geq b_2$ implies $b_1 = b_2$. A similar argument establishes that $a_1 = a_2$. Thus, two different intervals in $\{I_y\}$ are disjoint.

Sixth, establish that $\{I_y\}$ is countable. In each I_y , $\exists q \in \mathbb{Q}$ such that $q \in I_y$ by Theorem 100. Since I_y are disjoint, $q \in I_y$ and $q' \in I_{y'}$, for $y \neq y'$ implies $q \neq q'$. Hence there exists a one-to-one correspondence between the collection $\{I_y\}$ and a subset of the rational numbers. Thus, $\{I_y\}$ is countable by an argument similar to that in Example 77. ■

Figure 3.4.3 illustrates the theorem for the open set $O = O^1 \cup O^2$ where $O^1 = (-1, 0)$ and $O^2 = (\sqrt{2}, \infty)$. Part (a) of the figure illustrates steps 1 to 4. For example, take $y = -\frac{1}{4} \in O^1$. Then the supremum of the set of

upper interval endpoints around $-\frac{1}{4}$ contained in O^1 is $b_{-\frac{1}{4}} = 0$ and the infimum of the set of lower interval endpoints around $-\frac{1}{4}$ contained in O^1 is $a_{-\frac{1}{4}} = -1$ so that $I_{-\frac{1}{4}} = (-1, 0)$ which is just O^1 . Similarly take $y = \frac{3}{2}$. Then the supremum of the set of upper interval endpoints around $\frac{3}{2}$ contained in O^2 is $b_{\frac{3}{2}} = \infty$ and the infimum of the set of lower interval endpoints around $\frac{3}{2}$ contained in O^2 is $a_{\frac{3}{2}} = \sqrt{2}$ so that $I_{\frac{3}{2}} = (\sqrt{2}, \infty)$ which is just O^2 . Part (b) of the figure illustrates step 6, where the injection is finite (and hence countable).

Now we move on to closed sets.

Definition 109 $C \subset \mathbb{R}$ is **closed** if its complement (i.e. $\mathbb{R} \setminus C$) is open.

Example 110 $[0, 1] \subset \mathbb{R}$ is closed since its complement $\mathbb{R} \setminus [0, 1] = (-\infty, 0) \cup (1, \infty)$ is open since the union of open sets is open by Theorem 106. $(0, 1]$ is not closed since its complement, $(-\infty, 0) \cup [1, \infty)$, is not open. The singleton set $\{1\}$ is closed since its complement, $(-\infty, 1) \cup (1, \infty)$ is open. The set \mathbb{N} is closed since its complement, $(\cup_{n=1}^{\infty} (n-1, n)) \cup (-\infty, 0)$, is a countable union of open sets and hence by Theorem 106 is open.

There is another way to describe closed sets which uses cluster points.

Definition 111 A point $x \in \mathbb{R}$ is a **cluster point** of a subset $A \subset \mathbb{R}$ if any open ball around x intersects A at some point other than x itself (i.e. $(B_{\delta}(x) \setminus \{x\}) \cap A \neq \emptyset$).

Note that the point x may lie in A or not. A cluster point must have points of A sufficiently near to it as the next examples show.

Example 112 (i) Let $A = (0, 1]$. Then every point in the interval $[0, 1]$ is a cluster point of A . In particular, the point 0 is a cluster point since for any $\delta > 0$, $\exists y = \frac{\delta}{2} \in B_{\delta}(0)$ such that $B_{\delta}(0) \cap A \subset A$. (ii) Let $A = \{\frac{1}{n}, n \in \mathbb{N}\}$. Then 0 is the only cluster point of A . To see why, for any δ , just take $n_{\delta} = \frac{1}{\delta} + 1$, in which case for any $\delta > 0$, $\exists y = \frac{\delta}{1+\delta} \in B_{\delta}(0)$ such that $B_{\delta}(0) \cap A \subset A$. (iii) Let $A = \{0\} \cup (1, 2)$. Then $[1, 2]$ are the only cluster points of A since for any $\delta \in (0, 1)$, $B_{\delta}(0) \cap A = \emptyset$. (iv) \mathbb{N} has no cluster points for the same reason as (iii). (v) Let $A = \mathbb{Q}$. The set of cluster points of A is \mathbb{R} . This follows from Theorem 100 that between any two real numbers lies a rational. See Figure 3.4.4.

We next use Axiom 3 to prove a very important property of \mathbb{R} ; every nested sequence of closed intervals has a common point (and we can take that common point to either be the sup of the lower endpoints or the inf of the upper endpoints). First we must make that statement precise.

Example 113 *Returning to Example 97 where the open interval $S = (0, 1)$ and the closed interval $S' = [0, 1]$ are both bounded (and hence both possess a supremum by Axiom 3), only the closed interval S' contains its supremum of 1 (ie. has a maximum).*

Definition 114 *A set of intervals $\{I_n, n \in \mathbb{N}\}$ is **nested** if $I_1 \supset I_2 \supset \dots \supset I_n \supset I_{n+1} \supset \dots$*

Example 115 *A nested set of intervals does not necessarily have a common point (i.e. $\bigcap_{n=1}^{\infty} I_n = \emptyset$). For example, neither $I_n = (n, \infty)$ (so that $(1, \infty) \supset (2, \infty) \supset \dots$) nor $I_n = (0, \frac{1}{n})$ (so that $(0, 1) \supset (0, \frac{1}{2}) \supset \dots$) have common points. Why? It follows from the Archimedean Property 99 that for any $x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$. See Figure 3.4.5.*

Theorem 116 (Nested Intervals Property in \mathbb{R}) *If $\{I_n, n \in \mathbb{N}\}$ is a set of non-empty, closed, nested intervals in \mathbb{R} , then $\exists x \in \mathbb{R}$ such that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.*

Proof. Let $I_n = [a_n, b_n]$ with $a_n \leq b_n$. Since $I_1 \supset I_n$, then $b_1 \geq b_n \geq a_n$. Hence $\{a_n : n \in \mathbb{N}\}$ is bounded above and let α be its sup. To establish the claim, it is sufficient to show $\alpha \leq b_n, \forall n \in \mathbb{N}$. Suppose not. Then $\exists m \in \mathbb{N} \ni b_m < \alpha$. Since $\alpha = \sup\{a_n : n \in \mathbb{N}\}$, $\exists a_p > b_m$. Let $q = \max\{p, m\}$. Then $b_q \leq b_m < a_p \leq a_q$. But $b_q < a_q$ contradicts I_q is a non-empty interval. Thus $a_n \leq \alpha \leq b_n$ or $\alpha \in I_n, \forall n \in \mathbb{N}$. If I_n is not closed, then the last statement ($\alpha \in I_n$) doesn't necessarily hold. ■

See Figure 3.4.6. Note that the same arguments can be applied so that $\beta = \inf\{b_n | n \in \mathbb{N}\}$ is in every interval.

Example 117 *Let us return to Example 115. Instead of the open interval $I_n = (0, \frac{1}{n})$ consider the closed interval $I_n = [0, \frac{1}{n}]$ for which $\sup\{a_n | n \in \mathbb{N}\} = 0$. But it is clear that 0 is indeed in every nested interval. Another example of Theorem 116 may be $I_n = [-\frac{1}{n}, 1 + \frac{1}{n}]$. Obviously this is nested since $[-1, 2] \supset [-\frac{1}{2}, \frac{3}{2}] \supset [-\frac{1}{3}, \frac{4}{3}] \supset \dots$. In this case the $\sup\{a_n : n \in \mathbb{N}\} = \sup\{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\} = 0$, which is again in every interval. See Figure 3.4.7.*

We need the following important result to show that \mathbb{R} doesn't have any "holes" in it. ⁹In Section 4.2, we will show the precise meaning of this "absence of holes" property known as completeness. For now, one should simply recognize that to rule out holes, we need to draw out the implications of the Completeness Axiom 3. We do this through the next theorem.

Theorem 118 (Bolzano-Weierstrass) *Every bounded infinite subset $A \subset \mathbb{R}$ has a cluster point.*

Proof. (Sketch) If A is bounded, then there is a closed interval I such that $A \subset I$. Bisect I . There are infinitely many elements in at least one of the bisections. Denote such a bisection $I_1 \subset I$. Bisect I_1 . Again, there are infinitely many elements in at least one of the bisections. Denote such a bisection $I_2 \subset I_1$. By continuing this process we construct a set $\{I_n, n \in \mathbb{N}\}$ of non-empty, closed, nested intervals in \mathbb{R} . By Theorem 116, there is a point $x^* \in \bigcap_{n=1}^{\infty} I_n$, which is a cluster point of A . ■

Exercise 3.4.1 *Show that $x^* \in \bigcap_{n=1}^{\infty} I_n$ in Theorem 118 is a cluster point of A to finish the proof.*

In the proof we enclosed A in a closed interval $I = [a, b]$ and showed that any infinite subset of I has a cluster point. This special property of $[a, b]$ is called the *Bolzano-Weierstrass* property.

Definition 119 *A subset $A \subset \mathbb{R}$ has the **Bolzano-Weierstrass** property if every infinite subset of A has a cluster point belonging to A .*

We did not show that any infinite subset of A has a cluster point. The next example illustrates this.

Example 120 *Let $A = (a, b)$ with $b - a > 1$. Define $B = \{a + \frac{1}{n}, n \in \mathbb{N}\} \subset A$. The only cluster point of B is a , which doesn't belong to A . Thus open sets like (a, b) don't have the Bolzano-Weierstrass property. Boundedness is also important. Let $A = \mathbb{R}$. Then \mathbb{N} is an infinite subset of \mathbb{R} which does not have a cluster point.*

⁹In Section 4.2, we will show the precise meaning of the "absence of holes" property known as completeness. For now, one should simply recognize that to rule out holes, we need to draw out the implications of the completeness axiom. We do this through the Bolzano-Weierstrass Theorem.

We next present a necessary and sufficient condition for a subset of \mathbb{R} to have the Bolzano-Weierstrass property. This important result is known as the Heine-Borel theorem.¹⁰

Theorem 121 (Heine-Borel) *$A \subset \mathbb{R}$ has the Bolzano-Weierstrass property iff A is closed and bounded.*

Proof. (\Leftarrow) If A is finite, then A has the B-W property since infinite subsets of a finite set is a false statement.¹¹ Let A be infinite and let B be an infinite subset of A . Since B is bounded, it can be enclosed in a closed interval. Using the same procedure as in Theorem 118 we construct a cluster point x^* of B and hence also of A . Since A is closed, $x^* \in A$.

(\Rightarrow) “closedness”. Let x^* be a cluster point of A . Then for each $\delta = \frac{1}{n}$, $\exists x_n \in A$ such that $|x^* - x_n| < \frac{1}{n}$. The set $\{x_n\}_{n \in \mathbb{N}}$ is an infinite subset of A which has the B-W property so that $x^* \in A$.

“boundedness”. By contradiction. Suppose A is unbounded. Then for any n , $\exists x_n \in A$ such that $x_n > n$. Then $\{x_n\}$ is an infinite subset of A which doesn't have a cluster point (since $|x_{n+2} - x_n| > 1$ for all n). But this contradicts the B-W property. ■

We next use the Nested Intervals Property in \mathbb{R} (Theorem 116) to establish the uncountability of the set of real numbers.

Theorem 122 *$[0, 1]$ is uncountable.*

Proof. Suppose not. Then there exists a bijection $b : \mathbb{N} \rightarrow [0, 1]$. Then all elements from $[0, 1]$ can be numbered $\{x_1, x_2, \dots, x_n, \dots\}$. Divide $[0, 1]$ into three closed intervals: $I_1^1 = [0, \frac{1}{3}]$, $I_2^1 = [\frac{1}{3}, \frac{2}{3}]$, $I_3^1 = [\frac{2}{3}, 1]$. This implies x_1 is not contained in at least one of these three intervals.¹² WLOG, say it is I_1^1 . Divide I_1^1 into three closed intervals: $I_1^2 = [0, \frac{1}{9}]$, $I_2^2 = [\frac{1}{9}, \frac{2}{9}]$, $I_3^2 = [\frac{2}{9}, \frac{1}{3}]$. This implies $\exists I^2$ such that $x_2 \notin I^2$. Notice that $I^2 \subset I^1$ and that $x_1, x_2 \notin I^2$. In this way we can construct a sequence $\{I^n\}_{n=1}^\infty$ with the following properties: (i) I^n is closed; (ii) $I^1 \supset I^2 \supset \dots \supset I^n \supset \dots$ (i.e. nested intervals); and (iii) $x_i \notin I^n$, $\forall i = 1, \dots, n$. From (i) and (ii), Theorem 116 implies $\exists x^0 \in \bigcap_{n=1}^\infty I^n \subset [0, 1]$. So we have found a real number $x^0 \in [0, 1]$ which is different

¹⁰Those of you experienced readers may associate Heine-Borel with compactness. Since we wanted to keep this section simple, we'll put off the treatment of compactness until we work with more general metric spaces in Section 4.3.

¹¹And from a false statement, the implication is true by the truth table.

¹²It is possible x_1 is an element of 2 closed intervals (e.g. $x_1 = \frac{1}{3}$).

from any x_i , $i = 1, 2, \dots$. This contradicts our assumption that $\{x_1, x_2, \dots\}$ are all real numbers from $[0, 1]$. ■

While the above theorem establishes that $[0, 1]$ is uncountable (i.e. and hence really big in one sense), we next provide an example of an uncountable subset of $[0, 1]$ that is somehow small in another sense. This concrete example is known as the *Cantor set* and is constructed in the following way (see Figure 3.4.8). First, divide $[0, 1]$ into three "equal" parts: $[0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{2}{3})$, $[\frac{2}{3}, 1]$.¹³ Define $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ or equivalently $F_1 = [0, 1] \setminus A_1$ where $A_1 = (\frac{1}{3}, \frac{2}{3})$. That is, to construct F_1 we take out the center of $[0, 1]$. Second, divide each part of F_1 into three equal parts (giving us now 6 intervals). Define $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ or $F_2 = [0, 1] \setminus A_2$ where $A_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{3}{9}, \frac{6}{9}) \cup (\frac{7}{9}, \frac{8}{9})$. That is, to construct F_2 we take out the center of each of the two intervals in F_1 . By this process of removing the open "middle third" intervals, we construct F_n , $\forall n \in \mathbb{N}$. The Cantor set is just the intersection of the sets F_n . That is,

$$F = \bigcap_{n \in \mathbb{N}} F_n \left(\equiv [0, 1] \setminus \bigcup_{n \in \mathbb{N}} A_n \right).$$

The Cantor set has the following properties:

1. F is nonempty (by Theorem 116).
2. F is closed because it is the intersection of closed intervals F_n (by (iii) of Corollary ??, each F_n is closed because it is the union of finitely many closed intervals).
3. F doesn't contain any interval (a, b) with $a < b$ (by construction).
4. F is uncountable (by the same argument used in the proof of Theorem 122).

There are two important things to note about the Cantor set. First, while Theorem 108 says that any open set can be expressed as a countable union of open intervals, properties (1)-(4) of the Cantor set shows that there is no analogous result for closed sets. That is, a closed set may not in general be written as a countable union of closed intervals. In this sense, closed sets

¹³The sense in which we mean equal parts is that while the sets are different (some are closed, some open), they have the same distance between endpoints of $\frac{1}{3}$ (more formally, they have the same measure).

can have a more complicated structure than open sets. Second, property (4) above shows that even though F_n seem to be getting smaller and smaller in one sense (i.e. that it has many holes in it) in Figure 3.4.9, F is uncountable (and hence large in another sense).

3.5 Borel Sets

Since the intersection of a countable collection of open sets need not be open (e.g. Example 107), the collection of all open sets in \mathbb{R} is not a σ -algebra. By Theorem 87, however, there exists a smallest σ -algebra containing all open sets.

Definition 123 *The smallest σ -algebra generated by the collection of all open sets in \mathbb{R} , denoted \mathcal{B} , is called the **Borel σ -algebra in \mathbb{R} .***

Just as Example 83 showed in the case of algebras, even though \mathcal{B} is the smallest σ -algebra containing all open sets, it is bigger than just the collection of open sets. For example, we have to add back in singleton sets like those in Example 107 (i.e. the closed set $\{0\} = \bigcap_{n \in \mathbb{N}} (-\frac{1}{n}, \frac{1}{n})$) in order to keep it closed under countable intersection.¹⁴ In fact, almost any set that you can conceive of is contained in the Borel σ -algebra: open sets, closed sets, half open intervals $(a, b]$, sets of the form $\bigcap_{n \in \mathbb{N}} O_n$ with O_n open (which we saw is not necessarily open), sets of the form $\bigcup_{n \in \mathbb{N}} F_n$ with F_n closed (which we saw is not necessarily closed), and more. On the other hand, while finding a subset of \mathbb{R} which is not Borel requires a rather sophisticated construction (see p.?? of Jain and Gupta (1986)), the size of the collection of non-Borel sets is much bigger than the size of \mathcal{B} . Loosely speaking, \mathcal{B} is as thin in $\mathcal{P}(\mathbb{R})$ as \mathbb{N} is in \mathbb{R} (as we will see in Chapter 5).

Exercise 3.5.1 *Prove that the following sets in \mathbb{R} belong to \mathcal{B} : (i) any closed set; (ii) $(a, b]$.*

Borel sets can be generated by even smaller collections than all open sets as the next theorem shows.

¹⁴Recall in Example 83, for underlying set $X = \{a, b, c\}$, we showed that while $\mathcal{C}_4 = \{\emptyset, \{a\}, X\}$ was not an algebra (just as the collection of all open sets is not an algebra), we can create an algebra generated by $\{a\}$ (whose analogue is the Borel σ -algebra) which is just $\mathcal{C}_2 = \{\emptyset, \{a\}, \{b, c\}, X\} \subset \mathcal{P}(X)$ and is “bigger” in the sense of $\mathcal{C}_4 \subset \mathcal{C}_2$ (where $\{b, c\}$ plays the analogue of the other sets we have to add in).

Theorem 124 *The collection of all open rays $\{(a, \infty) : a \in \mathbb{R}\}$ generates \mathcal{B} .*

Proof. It is sufficient to show that any open set A can be constructed in terms of open rays. By Theorem 108, we know that $A = \cup_{n=1}^{\infty} I_n$ where I_n are disjoint open intervals. But $(a, b) = (a, \infty) \setminus [\cap_{n=1}^{\infty} (b - \frac{1}{n}, \infty)]$ with $a < b$. ■

Exercise 3.5.2 *Using the same idea, show that \mathcal{B} can be generated by the collection of all closed intervals $\{[a, b] : a, b \in \mathbb{R}, a < b\}$.*

Figures for Chapter 3

Figure 3.3.1: ub, lb, sup, inf

Figure 3.4.1: Open and Half-open unit intervals

Figure 3.4.2: Example where Countable Intersection of Open Intervals is
not Open

Figure 3.4.3a&b: Open Sets as a Countable Union of Disjoint Intervals

Figure 3.4.4: Examples of Cluster points

Figure 3.4.5: Examples of Nested Cells without a Common Point

Figure 3.4.6: Nested Cells Property

Figure 3.4.7: Example of a Common Point in Nested Cells

Figure 3.4.8 Cantor Set

3.6 Bibliography for Chapter 3

Sections 3.1 to 3.3 are based on Bartle (Sec 4 -6) and Royden (Ch 2., Sec 1 and 2).

3.7 End of Chapter Problems.

1. Let D be non-empty and let $f : D \rightarrow \mathbb{R}$ have bounded range. If D_0 is a non-empty subset of D , prove that

$$\inf\{f(x) : x \in D\} \leq \inf\{f(x) : x \in D_0\} \leq \sup\{f(x) : x \in D_0\} \leq \sup\{f(x) : x \in D\}$$

2. Let X and Y be non-empty sets and let $f : X \times Y \rightarrow \mathbb{R}$ have bounded range in \mathbb{R} . Let

$$f_1(x) = \sup\{f(x, y) : y \in Y\}, \quad f_2(y) = \sup\{f(x, y) : x \in X\}$$

Establish the Principle of Iterated Suprema:

$$\sup\{f(x, y) : x \in X, y \in Y\} = \sup\{f_1(x) : x \in X\} = \sup\{f_2(y) : y \in Y\}$$

(We sometimes express this as $\sup_{x,y} f(x, y) = \sup_x \sup_y f(x, y) = \sup_y \sup_x f(x, y)$).

3. Let f and f_1 be as in the preceding exercise and let

$$g_2(y) = \inf\{f(x, y) : x \in X\}.$$

Prove that

$$\sup\{g_2(y) : y \in Y\} \leq \inf\{f_1(x) : x \in X\}$$

(We sometimes express this as $\sup_y \inf_x f(x, y) \leq \inf_x \sup_y f(x, y)$).

Chapter 4

Metric Spaces

There are three basic theorems about continuous functions in the study of calculus (upon which most of calculus depends) that will prove extremely useful in your study of economics. They are the following:

1. *The Intermediate Value Theorem.* If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and if $r \in \mathbb{R}$ such that $f(a) \leq r \leq f(b)$, then $\exists c \in [a, b]$ such that $f(c) = r$.
2. *The Extreme Value Theorem.* If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then $\exists c \in [a, b]$ such that $f(x) \leq f(c), \forall x \in [a, b]$.
3. *The Uniform Continuity Theorem.* If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then given $\varepsilon > 0, \exists \delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon, \forall x_1, x_2 \in [a, b]$ for which $|x_1 - x_2| < \delta$.

These theorems are used in a number of places. The intermediate value theorem forms the basis for fixed point problems such as the existence of equilibrium. The extreme value theorem is useful since we often seek solutions to problems where we maximize a continuous objective function over a compact constraint set. The uniform continuity theorem is used to prove that every continuous function is integrable, which is important for proving properties of the value function in stochastic dynamic programming problems.

While we write these theorems in terms of real numbers, they can be formulated in more general spaces than \mathbb{R} . To this end, we will introduce (aliteratively) the 6 C's: convergence, closedness, completeness, compactness, connectedness, and continuity. In this chapter, we formulate these properties in terms of sequences. Each of the C properties uses some notion of distance.

For instance, convergence requires the distance between a limit point and elements in the sequence to eventually get smaller. Our goal in this chapter, is to consider theorems like those above but for any arbitrary set X . To do so however, requires X to be equipped with a distance function.

How will we proceed? First we will clarify what is meant by a distance function on an arbitrary set X . Then using the notion of convergence, which relies on distance, we will define closed sets in X . Then the collection of all closed (or by complementation open) sets is called a *topology on a set X* and it is the main building block in real analysis. That means properties such as continuity, compactness, and connectedness are defined directly or indirectly in terms of closed or open sets and for this reason are called topological properties. While there is an even more general way of defining a topology on X that doesn't use the notion of distance, we will wait until Chapter 7 to discuss it.

Definition 125 A *metric space* (X, d) is a nonempty set X of elements (called points) together with a function $d : X \times X \rightarrow \mathbb{R}$ such that $\forall x, y, z \in X$: (i) $d(x, y) \geq 0$; (ii) $d(x, y) = 0$ iff $x = y$; (iii) $d(x, y) = d(y, x)$; and (iv) $d(x, z) \leq d(x, y) + d(y, z)$. The function d is called a **metric**.

Example 126 We give three examples. First, let X be a set (e.g. $X = \{a, b, c, d\}$) and define a metric $d(x, y) = 0$ for $x = y$, and $d(x, y) = 1$ for $x \neq y$. This is called the “discrete metric”. It is easy to check that (X, d) is a metric space. Second, $(\mathbb{R}, |\cdot|)$, where d is simply the absolute value function and property (iv) is simply a statement of the triangle inequality. Thus, Chapter 3 should be seen as a special case of this chapter. Third, let X be the set of all continuous functions on $[a, b]$ and $d(f, g) = \sup \{|f(x) - g(x)|, x \in [a, b]\}$. In Chapter 6, we will see this as well as other metrics are valid metric spaces.

It should be emphasized that a metric space is not just the set of points X but the metric d as well. To see this, we introduce the notion of the cartesian product of metric spaces. Let (X, d_x) and (Y, d_y) be two metric spaces, then we can construct a metric d on $X \times Y$ from the metrics d_x and d_y . In fact, there are many metrics we can construct: $d_2(x, y) = \sqrt{(d_x(x_1, y_1))^2 + (d_y(x_2, y_2))^2}$ and $d_\infty(x, y) = \sup \{d_x(x_1, y_1), d_y(x_2, y_2)\}$.

Exercise 4.0.1 Show that d_2 and d_∞ are metrics in $X \times Y$.

Next, metrics provide us with the ability to measure the distance between two sets (if one of the sets is a singleton, then we can measure the distance of a point from a set).

Definition 127 Let $A \subset X$ and $B \subset X$. The **distance between sets A and B** is $d(A, B) = \inf\{d(x, y), x \in A, y \in B\}$.

We note that any subset of a metric space is a metric space itself.

Definition 128 If (X, d) is a metric space and $H \subset X$, then $(H, d|_r H)$ is also a metric space called the **subspace** of (X, d) .¹

Example 129 $([0, 1], |\cdot|)$ is a metric space which is a subspace of $(\mathbb{R}, |\cdot|)$.

In a metric space, we can extend the notion of open intervals in Definition (104).

Definition 130 For $x \in X$, we call the set $B_\delta(x) = \{y \in X : d(x, y) < \delta\}$ an **open ball with center x and radius δ** . In this case, G is open if $\forall x \in G, B_\delta(x) \subset G$.²

Don't assume that an open ball is an open set. We still don't know what an open set is. We will prove this in the next section. Also note that a ball is defined relative to the space X , so that if for example $X = \mathbb{N}$, then a ball of size $\delta = 1.5$ around 5 is just $\{4, 5, 6\}$. The next example shows that balls don't need to be "round". Their shape depends on their metric.

Example 131 In \mathbb{R}^2 , Figure 4.1 illustrates a ball with metric $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$, one with a Euclidean metric $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, and one with a sup metric $d_\infty(x, y) = \sup\{|x_1 - y_1|, |x_2 - y_2|\}$.

Before proceeding, we briefly mention some of the important results that you will see in this chapter. Here we extend the Heine Borel Theorem 121 of Chapter 3 to provide necessary and sufficient conditions for compactness in general metric spaces in Theorem 198. We also introduce the notion of a Banach space (a complete normed vector space) and for the first time give an example of an infinite dimensional Banach space. In many theorems that

¹Note that we restrict the metric function to the set H using Definition 56.

²Don't assume that an open ball is an open set. We will prove this in Section X.

follow, the dimensionality of a Banach space plays a crucial role. Another important set of results pertain to the properties of a continuous function on a connected domain (a generalization of the Intermediate Value Theorem is given in Theorem 254) as well as a continuous function on a compact domain (a generalization of the Extreme Value Theorem is given in Theorem 261 and the Uniform Continuity Theorem in general metric spaces is given in Theorem ??). Since many applications in economics result in correspondences, we spend considerable time on upper and lower hemicontinuous correspondences. Probably one of the most important theorems in economics is Berge's Theorem of the Maximum 295. The chapter concludes with a set of fixed point theorems that are useful in proving the existence of general equilibrium or existence of a solution to a dynamic programming problem.

4.1 Convergence

In this section we will build all the topological properties of a metric space in terms of convergent sequences (as an alternative to building upon open sets). In many cases, the sequence version (of definitions and theorems) is more convenient, easier to verify, and/or easier to picture.

Definition 132 *If X is any set, a **finite sequence** (or ordered \mathcal{N} -tuple) in X is a function $f : \Psi_{\mathcal{N}} \rightarrow X$ denoted $\langle x_n \rangle_{n=1}^{\mathcal{N}}$. An **infinite sequence** in X is a function $f : \mathbb{N} \rightarrow X$ denoted $\langle x_n \rangle_{n=1}^{\infty}$ (or $\langle x_n \rangle$ for short).*

When there is no misunderstanding, we assume all sequences are infinite unless otherwise noted. We use the $\langle x_n \rangle$ notation to reinforce the difference from $\{x_n | n \in \mathbb{N}\}$ since order matters for a sequence.

Example 133 *There are many ways of defining sequences. Consider the sequence of even numbers $\langle 2, 4, 6, \dots \rangle$. One way to list it is $\langle 2n \rangle_{n \in \mathbb{N}}$. Another way this is to specify an initial value x_1 and a rule for obtaining x_{n+1} from x_n . In the above case $x_1 = 2$ and $x_{n+1} = x_n + 2$, $n \in \mathbb{N}$.*

It is possible that while a sequence doesn't have some desired properties, but a subset of the sequence has the desired properties.

Definition 134 *A mapping $g : \mathbb{N} \rightarrow \mathbb{N}$ is **monotone** if $(n > m)$ implies $(g(n) > g(m))$. If $f : \mathbb{N} \rightarrow X$ is an (infinite) sequence, then h is an (infinite) **subsequence** of f if there is a monotone mapping $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $h = f \circ g$, denoted $\langle x_{g(n)} \rangle$.*

Example 135 Consider the sequence $f : \mathbb{N} \rightarrow \{-1, 1\}$ given by $\langle (-1)^n \rangle_{n \in \mathbb{N}}$. If $g(n) = 2n$ for $n \in \mathbb{N}$ (i.e. the even indices), then the subsequence $h = f \circ g$ is simply $\langle 1, 1, \dots \rangle$ while if $g(n) = 2n - 1$ (i.e. the odd indices), then the subsequence is $\langle -1, -1, \dots \rangle$. See Figure 4.1.1.

Definition 136 A sequence $\langle x_n \rangle$ from a metric space (X, d) **converges** to the point $\bar{x} \in X$ (or has \bar{x} as a **limit**), if given any $\delta > 0, \exists N$ (which may depend on δ) such that $d(\bar{x}, x_n) < \delta, \forall n \geq N(\delta)$. In geometric terms, this says that $\langle x_n \rangle$ converges to \bar{x} if every ball around \bar{x} contains all but a finite number of terms of the sequence. We write $\bar{x} = \lim x_i$ or $x_i \rightarrow \bar{x}$ to mean that \bar{x} is the limit of $\langle x_i \rangle$. If a sequence has no limit, we say it **diverges**.

Example 137 To see an example of a limit, consider the sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ given by $\langle (\frac{1}{n}) \rangle_{n \in \mathbb{N}}$. In this case the $\lim \langle (\frac{1}{n}) \rangle_{n \in \mathbb{N}} = 0$. To see why, notice that for any $\delta > 0$, it is possible to find an $N(\delta)$ such that $d(0, x_n) = |x_n| < \delta, \forall n \geq N(\delta)$. For instance, if $\delta = 1$, then $N(1) = 2$ (respects the strict inequality), if $\delta = \frac{1}{2}$, then $N(\frac{1}{2}) = 3$, etc. In general, let $N(\delta) = w(\frac{1}{\delta}) + 1$ where $w(x)$ denotes an operator which takes the whole part of the real number x . Such natural numbers always exist by the Archimedean Property (Theorem 99). See Figure 4.1.2.

Theorem 138 (Uniqueness of Limit Points) A sequence in (X, d) can have at most one limit.

Proof. Suppose, to the contrary, \bar{x}' and \bar{x}'' are limits of $\langle x_n \rangle$ and $\bar{x}' \neq \bar{x}''$. Let $B_\delta(\bar{x}') (= \{x \in X : d(x, \bar{x}') < \delta\})$ and $B_\delta(\bar{x}'')$ be disjoint open balls around \bar{x}' and \bar{x}'' , respectively.³ Furthermore, let $N', N'' \in \mathbb{N}$ be such that if $n \geq N'$ and $n \geq N''$, then $x_n \in B_\delta(\bar{x}')$ and $x_n \in B_\delta(\bar{x}'')$, respectively. Let $k = \max\{N', N''\}$. But then $x_k \in B_\delta(\bar{x}') \cap B_\delta(\bar{x}'')$, a contradiction. ■

Lemma 139 If $\langle x_n \rangle$ in (X, d) converges to $\bar{x} \in X$, then any subsequence $\langle x_{g(n)} \rangle$ also converges to \bar{x} .

Proof. By definition 136, $\exists N(\delta)$ such that $d(\bar{x}, x_n) < \delta, \forall n \geq N(\delta)$. Let $\langle x_{g(n)} \rangle$ be a subsequence of $\langle x_n \rangle$. Since $g(n) \geq n$, then $g(n) \geq N(\delta)$ in which case $d(\bar{x}, x_{g(n)}) < \delta$. ■

The next definition gives another notion of convergence to a point which is the sequential version of Definition 111.

³It is always possible to construct such disjoint balls. Just let $\delta = \frac{1}{4}d(\bar{x}', \bar{x}'')$.

Definition 140 A sequence $\langle x_n \rangle$ from a metric space (X, d) has a **cluster point** $x^* \in X$ if given any $\delta > 0$ and given any $N, \exists n \geq N$ such that $d(x^*, x_n) < \delta$. In geometric terms, this says that x^* is a cluster point of $\langle x_n \rangle$ if each ball around x^* contains infinitely many terms of the sequence.

Thus if $\bar{x} = \lim \langle x_n \rangle$, then it is a cluster point.⁴ However, if x^* is a cluster point, it need not be a limit. To see this, note that the key difference between Definitions 136 and 140 lie in what terms in the sequence qualify as a limit or cluster point. If \bar{x} is a limit point we know $\exists N(\delta)$ for which $d(\bar{x}, x_n) < \delta$ for $n \geq N(\delta)$. For a cluster point, given N , it is sufficient to find just one term in the sequence x_n sufficiently far out that satisfies $d(x^*, x_n) < \delta$. But then just take n in definition 140 as $n = \max\{N(\delta), N\}$. To see this, consider the next example.

Example 141 Consider the sequence $\langle (-1)^n \rangle_{n \in \mathbb{N}}$ from Example 135. This sequence has no limit point but two cluster points. To see why, notice that the only candidate limit points are $\{-1, 1\}$. Consider $\bar{x} = 1$. For all $\delta \in (0, 1)$, $d(1, x_i) = |1 - x_i| > \delta$ for any odd index $i = 2n - 1, n \in \mathbb{N}$. A similar argument holds for $\bar{x} = -1$. To see why $x^* = 1$ satisfies the definition of a cluster point, notice that for any N , there exists $i = 2N + 1$ (an odd index) such that for any $\delta > 0$, $d(1, x_i) < \delta$. For this particular sequence, there are actually an infinite number of such indices. See Figure 4.1.1.

Example 141 provides a sequence which does not have a limit point (and hence the assumption of Lemma 139 does not apply). However, it is easy to see that there is a subsequence $\langle (-1)^{g(n)} \rangle_{n \in \mathbb{N}}$ of odd indices that has a limit point (which is one of the cluster points of the original sequence). The following theorem applies to such cases.

Lemma 142 x^* is a cluster point of $\langle x_n \rangle$ iff there exists a subsequence $\langle x_{n_k} \rangle \subset \langle x_n \rangle$ such that $\langle x_{n_k} \rangle \rightarrow x^*$ as $k \rightarrow \infty$.

Proof. (\Rightarrow) If x^* is a cluster point, then $\forall \frac{1}{k} > 0, \exists x_{n_k}$ such that $d(x^*, x_{n_k}) < \frac{1}{k}$ for any N . (\Leftarrow) is trivial. ■

⁴One shouldn't be confused between cluster points for a set and for a sequence. For instance, singletons like $\{1\}$ do not have cluster points, whereas the constant sequence $\langle 1, 1, 1, \dots \rangle$ does (which is 1).

Definition 143 Let (X, d) be a metric space and $A \subset X$. A is **closed** if any convergent sequence of elements from A has its limit point in A .

Theorem 144 (Closed Sets Properties) (i) \emptyset and X are closed. (ii) The intersection of any collection of closed sets in X is closed. (iii) The union of any finite collection of closed sets in X is closed.

Proof. (i) Trivial.

(ii) Let $A = \bigcap_{i \in \Lambda} A_i$ where $A_i \subset X$ is closed $\forall i \in \Lambda$, which is any index set. Take any convergent sequence from A and show its limit point is in A as well. That is, let $\langle x_n \rangle \subset A$ and $\langle x_n \rangle \rightarrow \bar{x}$. Then $\langle x_n \rangle \subset A_i \forall i \in \Lambda$ and because A_i is closed $\langle x_n \rangle \rightarrow \bar{x}$ and $\bar{x} \in A_i \forall i \in \Lambda$ implies $\bar{x} \in \bigcap_{i \in \Lambda} A_i$.

(iii) Let $A = \bigcup_{i=1}^n A_i$ where $A_i \subset X$ is closed $\forall i \in \{1, \dots, n\}$. Again, take any convergent sequence from A and show its limit point is in A as well. In particular, let $\langle x_n \rangle \subset A$ and $\langle x_n \rangle \rightarrow \bar{x}$. There exists A_j containing infinitely many elements of $\langle x_n \rangle$ (i.e. A_j contains a subset $\langle x_{n_k} \rangle$). By lemma 139, $\langle x_{n_k} \rangle \rightarrow \bar{x}$ and because $\langle x_{n_k} \rangle \subset A_j$ and A_j is closed, then $\bar{x} \in A_j$ implies $\bar{x} \in A = \bigcup_{i=1}^n A_i$. ■

Example 145 Property (iii) of Theorem 144 does not necessarily hold for infinite union. Consider the following counterexample. Let $F_n = [-1, -\frac{1}{n}] \cup [\frac{1}{n}, 1]$. Then $\bigcup_{n=1}^{\infty} F_n = [-1, 0) \cup (0, 1]$. See Figure 4.1.3.

Closed sets can also be described in terms of cluster points.

Theorem 146 A subset of X is closed iff it contains all its cluster points.

Proof. ⁵(\Rightarrow) By contradiction. Let x be a cluster point of a closed set A and let $x \notin A$. Then $x \in X \setminus A$. Because $X \setminus A$ is open, there exists an open ball $B_{\delta_x}(x)$ such that $B_{\delta_x}(x) \subset X \setminus A$. Thus $B_{\delta_x}(x)$ is a neighborhood of x having empty intersection with A . This contradicts the assumption that x is a cluster point of A .

(\Leftarrow) Let $x \in X \setminus A$. Then x is not a cluster point of A since A contains all its cluster points by assumption. Then there exists an open ball $B_{\delta_x}(x)$ such that $A \cap B_{\delta_x}(x) = \emptyset$. This implies $B_{\delta_x}(x) \subset X \setminus A$ or that $X \setminus A$ is open, in which case A is closed. ■

⁵From Munkres Theorem 6.6, p. 97.

Exercise 4.1.1 Explain why a singleton set $\{x\}$ is consistent with Theorem 146.

We now introduce another topological notion that permits us to characterize closed sets in other terms.

Definition 147 Given a set $A \subset X$, the union of all its points and all its cluster points is called the **closure** of A , denoted \overline{A} (i.e. $\overline{A} = A \cup A'$ where A' is the set of all cluster points of A).

Notice that \overline{A} is not a partition since A and A' are not necessarily disjoint. Take (iii) of Example 112 where $A = \{0\} \cup (1, 2)$, in which case $A' = [1, 2]$ and $\overline{A} = \{0\} \cup [1, 2]$.

As an exercise, prove the following theorems.

Theorem 148 Let $A \subset X$. $x \in \overline{A}$ iff any open ball around x has a non-empty intersection with A .

Theorem 149 The closure of A is the intersection of all closed sets containing A .

Theorem 150 A is closed iff $A = \overline{A}$.

Exercise 4.1.2 Prove that $A \subset \overline{A}$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Give an example to show that $\overline{A \cap B} = \overline{A} \cap \overline{B}$ may not hold.

Example 151 (i) $A = \{2, 3\}$. Then $\overline{A} = \{2, 3\}$. (ii) $A = \mathbb{N}$. Then $\overline{A} = \mathbb{N}$. (iii) $A = (0, 1]$. Then $\overline{A} = [0, 1]$. (iv) $A = \{x \in \mathbb{Q} : x \in (0, 1)\}$. Then $\overline{A} = [0, 1]$.

Intuitively, one would expect that a point x lies in the closure of A if there is a sequence of points in A converging to x . This is not necessarily true in a general topological space, but it is true in a metric space as the next lemma shows.

Lemma 152 Let (X, d) be a metric space and $A \subset X$. Then $x \in \overline{A}$ iff it is a limit point of a sequence $\langle x_n \rangle$ of points from A (i.e. $\exists \langle x_n \rangle \subset A$ such that $\langle x_n \rangle \rightarrow \overline{x}$).

Proof. (\Rightarrow) Take any $x \in \overline{A}$. By Theorem 148, $\forall \delta = \frac{1}{n} > 0$, $\exists x_n \in A$ such that $x_n \in B_{\frac{1}{n}}(x)$. Hence this sequence $\langle x_n \rangle \rightarrow x$ (a limit point). (\Leftarrow) If $\langle x_n \rangle \rightarrow x$ such that $\langle x_n \rangle \subset A$, then in every open ball around x there is x_n (actually, infinitely many of them) inside this ball. Then by Theorem 148 $x \in \overline{A}$. ■

Exercise 4.1.3 (i) Show that if A is closed and $d(x, A) = 0$, then $x \in A$. Does (i) hold without assuming closedness of A ? (ii) Show that if A is closed and $x_0 \notin A$, then $d(x_0, A) > 0$.

In the previous Example 151, we see that in some cases the closure of a set is: the set itself (i); "brings" finitely many new points to the set (iii); or brings uncountably many points (iv). This leads us to the notion of density.

Definition 153 Given the metric space (X, d) , a subset $A \subset X$ is **dense** in X if $\overline{A} = X$.

Example 154 To see that \mathbb{Q} is dense in \mathbb{R} , we know that in any ball around $x \in \mathbb{R}$ there is a rational number. Hence by Theorem 148 x is from $\overline{\mathbb{Q}}$. Thus we have $\mathbb{R} \subset \overline{\mathbb{Q}}$. Obviously, $\mathbb{R} \supset \overline{\mathbb{Q}}$ as well, so $\mathbb{R} = \overline{\mathbb{Q}}$. A similar argument establishes that the set of irrationals is dense in \mathbb{R} .

Intuitively, if A is dense in X then for any $x \in X$, there exists a point in A that is sufficiently close to (or approximates) x . From the previous example, since \mathbb{Q} is dense in \mathbb{R} , any real number can be approximated by a rational number which is countable. More importantly for applied economists, we might take X to be the set of continuous functions and A the set of polynomials with rational coefficients which is again countable. Then, provided the set of such polynomials is dense in the set of all continuous functions, working with polynomials will yield a good approximation to the continuous function we are interested in.

Definition 155 A metric space (X, d) is **separable** if it contains a dense subset that is countable.

Example 156 $(\mathbb{R}, |\cdot|)$ is separable since \mathbb{Q} is a countable dense subset of \mathbb{R} .

So far in a general metric space we have dealt only with closed sets. Now we can introduce open sets as follows.

Definition 157 A set $A \subset X$ is **open** if its complement is closed.

Exercise 4.1.4 Show that an open ball is an open set.

Example 158 Let $A = \{(0, 1) \times \{2\}\} = \{(x, y) : 0 < x < 1, y = 2\} \subset \mathbb{R}^2$ equipped with d_2 . A is not open since no matter how small δ is, there exist $y' = 2 \pm \frac{\delta}{2}$ such that $(x, y') \in B_\delta((x, 2))$ is not contained in A . See Figure 4.1.4.

We could have proven the properties of open sets as we did in Theorem 106, but we will not repeat it. Here we will simply mention a few concepts that will be useful. The first concept is that of a neighborhood.

Definition 159 A **neighborhood of** $x \in X$ is an open set containing x .

Sometimes it is more convenient to use the concept of a neighborhood of x rather than an open ball around x , but you should realize that these two concepts are equivalent since an open ball $B_\varepsilon(x)$ is a neighborhood of x and conversely, if V_x is a neighborhood of x , then there is a $B_\varepsilon(x) \subset V_x$. See Figure 4.1.5.

There is another way to describe closed sets which uses boundary points.

Definition 160 A point $x \in X$ is a **boundary point** of A if every open ball around x contains points in A and in $X \setminus A$ (i.e. $(B_\delta(x) \cap A) \cap (B_\delta(x) \cap (X \setminus A)) \neq \emptyset$). A point $x \in X$ is an **interior point** of A if $\exists B_\delta(x) \subset A$. See Figure 4.1.6.

Note that a boundary point need not be contained in the set. For example, the boundary points of $(0, 1]$ are 0 and 1.

Example 161 The set of boundary points of \mathbb{Q} is \mathbb{R} since in any open ball around a rational number there are other rationals and irrationals by Theorems 100 and 102.

The next theorem provides an alternative characterization of a closed set.

Theorem 162 A set $A \subset X$ is closed iff it contains its boundary points.

Proof. (\Rightarrow) Suppose A is closed and x is a boundary point. If $x \notin A$, then $x \in X \setminus A$ (which is open), contrary to x being a boundary point. (\Leftarrow) Suppose A contains all its boundary points. If $y \notin A$, then $\exists B_\delta(y)$ a proper subset of $X \setminus A$. Since this is true $\forall y \notin A$, $X \setminus A$ is open so A is closed. ■

Unlike properties like closedness and openness, boundedness is defined relative to the distance measure and hence is a metric property rather than a topological property.

Definition 163 Given (X, d) , $A \subset X$ is **bounded** if $\exists M > 0$ such that $d(x, y) \leq M, \forall x, y \in X$.

Boundedness cannot be defined only in terms of open sets. It requires the notion of distance. Thus it is not a topological property.

Theorem 164 A convergent sequence in the metric space (X, d) is bounded.

Proof. Taking $\delta = 1$, we know by Definition 136, $\exists N(1)$ such that $|x_n - \bar{x}| < 1, \forall n \geq N(1)$.⁶ By the triangle inequality, we know $|x_n| = |x_n - \bar{x} + \bar{x}| \leq |x_n - \bar{x}| + |\bar{x}| < 1 + |\bar{x}|, \forall n \geq N(1)$. Since there are a finite number of indices $n < N(1)$, then we set $M = \sup\{|x_1|, |x_2|, \dots, |x_{N(1)-1}|, |\bar{x}| + 1\}$. Hence, $|x_n| \leq M, \forall n \in \mathbb{N}$, so that $\langle x_i \rangle$ is bounded. ■

4.1.1 Convergence of functions

While we will focus on convergence of functions in Chapter 6, it will be necessary for some results in the upcoming sections to introduce a form of functional convergence. A sequence of functions is simply a sequence whose elements $f_n(x)$ contain two variables, n and x , where n indicates the order in the sequence and x is the variable of a function. For example, $\langle f_n(x) \rangle = \langle x^n \rangle = \langle x, x^2, x^3, \dots \rangle$ for $x \in [0, 1]$.

What does it mean for a sequence of functions to be convergent? There are basically two different answers to this question. If we work in a metric space whose elements are functions themselves with a certain metric, then convergence of functions is nothing other than convergence of elements (the element being a function) with respect to the given metric. We will deal with this type of convergence in Chapter 6 on function spaces. The second

⁶Since the sequence converges, we are free to choose any $\varepsilon > 0$. Here we simply choose $\varepsilon = 1$.

approach is to take any set X along with a metric space (Y, d_Y) and let $f_n : X \rightarrow Y$ for all n . Fix $x_0 \in X$. Then $\langle f_n(x_0) \rangle$ is a sequence of elements (the element being a point) in Y . If this sequence is convergent, then it converges to a certain point y_0 (i.e. $\langle f_n(x_0) \rangle \rightarrow y_0 = f(x_0)$). This leads to the following definition.

Definition 165 Given any set X and a metric space (Y, d_Y) , let $\langle f_n \rangle$ be a sequence of functions from X to Y . The sequence $\langle f_n \rangle$ is said to **converge pointwise** to a function $f : X \rightarrow Y$ if for every $x_0 \in X$, $\lim_{n \rightarrow \infty} f_n(x_0) = f(x_0)$. We call f a **pointwise limit** of $\langle f_n \rangle$. In other words, $\langle f_n \rangle$ converges pointwise to f on X if $\forall x_0 \in X$ and $\forall \varepsilon > 0$, $\exists N(x_0, \varepsilon)$ such that $\forall n > N(x_0, \varepsilon)$ we have $d_Y(f_n(x_0), f(x_0)) < \varepsilon$.

Notice that if x_0 is fixed, then $\langle f_n(x_0) \rangle$ is simply a sequence of elements in the metric space (Y, d_Y) .

Example 166 Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ given by $f_n(x) = \frac{x}{n}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 0$. Thus, this example is a simple generalization of Example 137. Then $\langle f_n \rangle$ converges pointwise to f since we can always find a natural number $N(x, \varepsilon) = w\left(\left|\frac{x}{\varepsilon}\right|\right) + 1$ by the Archimedean Property. See Figure 4.1.7.

Example 167 Let $f_n : [0, 1] \rightarrow \mathbb{R}$ given by $f_n(x) = x^n$ and $f : [0, 1] \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}.$$

It is clear that when $x = 1$, then $f_n(x) = 1^n = 1 = f(x)$ so that $f_n(1) \rightarrow 1$ trivially. To see that for $x \in [0, 1)$, $f_n(x) \rightarrow f(x)$ note that if we write $x = \frac{1}{1+a}$ with $a > 0$ then we can use Bernoulli's inequality that $(1+a)^n \geq 1+na$, then $0 < x^n = \left(\frac{1}{1+a}\right)^n \leq \frac{1}{1+na} < \frac{1}{na}$ so that we can take $N(x, \varepsilon) = w\left(\frac{1}{a\varepsilon}\right) + 1$. See Figure 4.1.8.

Notice that the rate of convergence $N(x_0, \varepsilon)$ can be very different for each x_0 . In Example 166, the rate is very low for very large x (we say the rate of convergence is smaller the larger is N). However, if we restrict the domain for f_n , say $f_n : [0, 2] \rightarrow \mathbb{R}$, then the smallest possible rate for a given ε is $N(2, \varepsilon) = w\left(\left|\frac{2}{\varepsilon}\right|\right) + 1$. If it is possible, for a given ε , to find the rate independently of x , then we call this type of convergence uniform.

Definition 168 Given X and a metric space (Y, d_Y) , let $\langle f_n \rangle$ be a sequence of functions from X to Y . The sequence $\langle f_n \rangle$ is said to **converge uniformly** to a function $f : X \rightarrow Y$ if $\forall \varepsilon > 0, \exists N(\varepsilon)$ such that $\forall n > N(\varepsilon)$ we have $d_Y(f_n(x), f(x)) < \varepsilon, \forall x \in X$.

It is apparent from the definition that uniform convergence implies pointwise convergence, but as Examples 166 and 167 show the converse does not necessarily hold (i.e. the above two sequences of functions do not converge uniformly but do converge pointwise). In the first case, $\forall \varepsilon > 0$ and $\forall n \in \mathbb{N}$, $\exists x_0 \in \mathbb{R}$ such that $x_0 > \frac{\varepsilon}{n}$ by the Archimedean property \mathbb{X} so that $\frac{x_0}{n} > \varepsilon$. Similarly, in the second example, $\forall \varepsilon \in (0, 1)$ and $\forall n \in \mathbb{N}$, we have $1 > \varepsilon^{\frac{1}{n}} > 0$. Then $\exists x_0 \in (0, 1)$ such that $1 > x_0 > \varepsilon^{\frac{1}{n}} > 0$ in which case $x_0^n > \varepsilon$. On the other hand, if we restrict the domain of the first example to $[-1, 1]$ (or for that matter any bounded set) f_n is uniformly convergent since for any ε we can take $N = \frac{1}{\varepsilon} + 1$.

4.2 Completeness

The completeness of a metric space is a very important property for problem solving. For instance, to prove the existence of the solution of a problem, we usually manage to find the solution of an approximate problem. That is, we construct a sequence of solutions that are getting closer and closer to one another via the method of successive approximations. But for this method to work, we need a guarantee that the limit point *exists*. If the space is complete, then the limit of this sequence *exists* and is the solution of the original problem. For this reason, we turn to establishing when a given space is complete.

Definition 169 A sequence $\langle x_n \rangle$ from a metric space (X, d) is a **Cauchy sequence** if given $\delta > 0, \exists N(\delta)$ such that $d(x_m, x_n) < \delta, \forall m, n \geq N(\delta)$.

Note that if $\langle x_n \rangle$ is convergent, then there is a limit point \bar{x} to which elements of $\langle x_n \rangle$ eventually approach. If $\langle x_n \rangle$ is Cauchy, then elements of $\langle x_n \rangle$ eventually approach a point which may or may not exist. Hence all Cauchy sequences can be divided into two different classes: those for which $\exists \bar{x}$ such that $\langle x_n \rangle \rightarrow \bar{x}$ (i.e. convergent Cauchy sequences); and those for which $\nexists \bar{x}$ such that $\langle x_n \rangle \rightarrow \bar{x}$ (i.e. nonconvergent Cauchy sequences).

Example 170 Suppose we did not know that there existed a limit in example 137 where $\langle (\frac{1}{n}) \rangle_{n \in \mathbb{N}}$ in $(\mathbb{R}, |\cdot|)$. We can however, establish that this sequence of real numbers is a Cauchy sequence and hence has a limit. Let $m, n \geq N(\delta)$ and without loss of generality let $m \leq n$. Then $d(s_n, s_m) = |\frac{1}{m} - \frac{1}{n}| < \frac{1}{m}$. Hence a sufficient condition for this sequence to be Cauchy for any $\delta > 0$ is that $N(\varepsilon) = w(\frac{1}{\delta}) + 1$.

Example 171 Consider the metric space (X, d) with $X = (0, 1]$ and $d = |x|$. Then by Example 170, we've established that $\langle (\frac{1}{n}) \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence that converges (in \mathbb{R}) to a limit $0 \notin X$.

We now list some results that are not so useful in and of themselves but will be used repeatedly to prove important theorems in the next few sections.

Lemma 172 Given (X, d) , if $\langle x_n \rangle$ converges, then $\langle x_n \rangle$ is a Cauchy sequence.

Proof. Let $\bar{x} = \lim \langle x_n \rangle$. Then given $\delta > 0, \exists N(\frac{\delta}{2})$ such that if $n \geq N(\frac{\delta}{2})$, then $d(\bar{x}, x_n) < \frac{\delta}{2}$. Thus if $n, m \geq N(\frac{\delta}{2})$, then

$$d(x_m, x_n) \leq d(x_m, \bar{x}) + d(\bar{x}, x_n) < \frac{\delta}{2} + \frac{\delta}{2}.$$

■

Lemma 173 If a subsequence $\langle x_{g(n)} \rangle$ of a Cauchy sequence $\langle x_n \rangle$ converges to \bar{x} , then $\langle x_n \rangle$ also converges to \bar{x} .

Exercise 4.2.1 Prove Lemma 173.

Lemma 174 A Cauchy sequence in (X, d) is bounded.

Exercise 4.2.2 Prove Lemma 174. It is similar to Lemma 164 in the preceding section.

The converse of Lemma 172 is not necessarily true. Those spaces for which the converse of Lemma 172 is true are called complete.

Definition 175 If (X, d) has the property that every Cauchy sequence converges to some point in the metric space, then (X, d) is **complete**.

Establishing a metric space is complete is a difficult task since we must show that *every* Cauchy sequence converges. In fact, due to Lemma 173 we can (somewhat) weaken this definition, which gives us the following lemma.

Lemma 176 *(X, d) is complete if every Cauchy Sequence has a convergent subsequence.*

Proof. It is sufficient to show that if $\langle x_n \rangle$ is a Cauchy sequence that has a subsequence $\langle x_{g(n)} \rangle$ which converges to \bar{x} , then $\langle x_n \rangle$ converges to \bar{x} . Since $\langle x_n \rangle$ is a Cauchy sequence, given $\delta > 0$, we can choose $N(\frac{\delta}{2})$ large enough such that $d(x_m, x_n) < \frac{\delta}{2}$, $\forall m, n \geq N(\frac{\delta}{2})$ by Definition 169. Since $\langle x_{g(n)} \rangle$ is a convergent subsequence, given $\delta > 0$, we can choose $N(\frac{\delta}{2})$ large enough such that $d(x_{g(n)}, \bar{x}) < \frac{\delta}{2}$, $\forall g(n) \geq N(\frac{\delta}{2})$ by Definition 136. Combining these two facts and using (iv) of Definition 125, $d(x_n, \bar{x}) \leq d(x_n, x_{g(n)}) + d(x_{g(n)}, \bar{x}) < \delta$. ■

Another useful fact is that if we know a space is complete, then we know a closed subspace is complete.

Theorem 177 *A closed subset of a complete metric space is complete.*

Proof. Any Cauchy sequence in the closed subset is a Cauchy sequence in the metric space. Since the metric space is complete, it is convergent. Since the subset is closed, the limit also must be from this set. ■

Establishing that a metric space is not complete is an easier task since we must only show that *one* Cauchy sequence does not converge to a point in the space. Just take $((0, 1], |\cdot|)$ in Example 171 since the limit of $\langle (\frac{1}{n}) \rangle$ is 0 which is not contained in $(0, 1]$.

Example 178 *Consider the sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ given by $\langle (1 + \frac{1}{n})^n \rangle_{n \in \mathbb{N}}$. It can be shown that this sequence is increasing and bounded above. Then by the Monotone Convergence Theorem 324, which is proven in the End of Chapter Exercises, this sequence converges in $(\mathbb{R}, |\cdot|)$. The limit of this sequence is called the Euler number e ($e = 2.71828\dots$), which is irrational. But then $(\mathbb{Q}, |\cdot|)$ is not complete; the sequence $\langle (1 + \frac{1}{n})^n \rangle_{n \in \mathbb{N}} \subset \mathbb{Q}$ is Cauchy (because it is convergent in \mathbb{R}) but is not convergent in \mathbb{Q} .*

Example 179 *While \mathbb{Q} is not complete, \mathbb{N} is complete because the only Cauchy sequences in \mathbb{N} are constant sequences (e.g. $\langle 1, 1, 1, \dots \rangle$), which are also convergent.*

We next take up the important question of completeness of $(\mathbb{R}, |\cdot|)$. This takes some work.

Theorem 180 (Bolzano-Weierstrass for Sequences) *A bounded sequence in \mathbb{R} has a convergent subsequence.*

Proof. Let $A = \{x_n\}$ be bounded. If there are only a finite number of distinct values in the sequence, then at least one of these values must occur infinitely often. If we define a subsequence $\{x_{g(n)}\}$ of $\{x_n\}$ by selecting this element each time it appears we obtain a convergent (constant) subsequence.

If the sequence $\{x_n\}$ contains infinitely many distinct values, then $A = \{x_n\}$ is infinite and bounded. By the Bolzano-Weierstrass Theorem 118 for sets (which rested upon the Nested Cells Property, which in turn rested upon the Completeness Axiom), there is a cluster point x^* of $A = \{x_n\}$. Then by Theorem 142 there is a subsequence $\{x_{g(n)}\} \rightarrow x^*$. ■

Theorem 181 (Cauchy Convergence Criterion) *A sequence in \mathbb{R} is convergent iff it is a Cauchy sequence.*

Proof. (\Rightarrow) is true in any metric space by Lemma 172.

(\Leftarrow) Let $\{x_n\}$ be a Cauchy sequence in \mathbb{R} . Then it is bounded by Lemma 174 and by Theorem 180 there is a convergent subsequence $\{x_{n_k}\} \rightarrow \bar{x}$. Then by Lemma 173, the whole sequence $\{x_n\}$ converges to \bar{x} . ■

Hence, since completeness requires that any Cauchy sequence converges, we know from Theorem 181 that $(\mathbb{R}, |\cdot|)$ is complete.

4.2.1 Completion of a metric space.

Every metric space can be made complete. The idea is a simple one. Let (X, d) be a metric space that is not complete. Let $\mathcal{CS}[X]$ be the set of all Cauchy sequences on the incomplete metric space and let $\{a_n\}, \{b_n\} \in \mathcal{CS}[X]$. Define (as in Definition 26) the equivalence relation " \sim " by $\{a_n\} \sim \{b_n\}$ iff $\lim_{n \rightarrow \infty} d(a_n, b_n) = 0$. This relation forms a partition of $\mathcal{CS}[X]$ where in every equivalence class there are all sequences which have the same limit. Let X^* be the set of all equivalence classes of $\mathcal{CS}[X]$. Then X^* with the metric $\hat{d}([\{a_n\}], [\{b_n\}]) = \lim_{n \rightarrow \infty} d(a_n, b_n)$ is a complete metric space.

Example 182 *Reconsider Example 171. The completion of $((0, 1), |\cdot|)$ is $([0, 1], |\cdot|)$. Notice that we added two Cauchy sequences $\langle \frac{1}{n} \rangle$ and $\langle 1 - \frac{1}{n} \rangle$.*

Next we demonstrate the process of completion of a metric space $(\mathbb{Q}, |\cdot|)$, which we know by example 178 is not complete since $\langle (1 + \frac{1}{n})^n \rangle$ is a non-convergent Cauchy sequence (in \mathbb{Q}). Let $\mathcal{CS}(\mathbb{Q})$ be the set of all Cauchy sequences. An equivalence relation, defined in 26, partitions $\mathcal{CS}(\mathbb{Q})$ into classes like those shown in Figure 4.2.1: classes of convergent Cauchy sequences such as $\langle 1 + \frac{1}{n} \rangle$ and $\langle 1 - \frac{1}{n} \rangle$ (which converges to the rational number 1) and classes of non-convergent (in \mathbb{Q}) like $\langle (1 + \frac{1}{n})^n \rangle$ (which converges to e which is not in \mathbb{Q}). Loosely speaking, we can then assign the number 1 to the class of convergent Cauchy sequences and can assign e to the non-convergent Cauchy sequence. How can we compare two metric spaces with completely different objects (e.g. one containing classes of Cauchy sequences and the other containing real numbers)?

Definition 183 *Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $f : X \rightarrow Y$ have the following property*

$$d_X(x, y) = d_Y(f(x), f(y)). \quad (4.1)$$

*A function f having this property is called an **isometry**.*

By (4.1) it is clear that an isometry is always an injection. If f is also a surjection, then it is a bijection and in this case we say that (X, d_X) and (Y, d_Y) are *isometric*. Two isometric spaces might have completely different objects, but due to (4.1) and the fact that f is a bijection, they are exact replicas of one another. Their objects just have different names.

Getting back to our example, consider the spaces $(\mathbb{R}, |\cdot|)$ and $(\mathcal{CS}(\mathbb{Q}), \widehat{d})$. Let $f : \mathbb{R} \rightarrow \mathcal{CS}(\mathbb{Q})$ given by $f(x) = \{\langle x_n \rangle^1, \langle x_n \rangle^2, \dots\}$ where $\langle x_n \rangle^1 \rightarrow x$, $\langle x_n \rangle^2 \rightarrow x$, etc. are Cauchy sequences of one class converging to x (e.g. $\langle x_n \rangle^1 = \langle 1 + \frac{1}{n} \rangle$ and $\langle x_n \rangle^2 = \langle 1 - \frac{1}{n} \rangle$ which converge to 1). f is a surjection because $(\mathbb{R}, |\cdot|)$ is complete. One can show that f is an isometry. Thus, these two metric spaces are isometric. The above construction implies the fact that for every real number $x \in \mathbb{R}$ there exists a sequence $\langle x_n \rangle$ of rational numbers converging to x (i.e. $\lim x_n = x$ where $x_n \in \mathbb{Q}$).

4.3 Compactness

When we listed the three important theorems at the beginning of this chapter, there was a common assumption; the domain of f was taken to be the closed interval $[a, b]$. What properties of $[a, b]$ guarantee the validity of these theorems? What properties of the domain of f are necessary for the validity of comparable theorems in more general metric spaces? In more general metric spaces, closed intervals may not even be defined. If we replaced the closed interval above with a closed ball, would the results continue to hold? As we will see later in the Chapter, they may not.

In Chapter 3 we showed (Theorem 121) that $[a, b]$ has the Bolzano-Weierstrass property such that any sequence of elements of $[a, b]$ has a subsequence that converges to a point in $[a, b]$. As we will see, this property can also be defined for a subset A of a general metric space (X, d) to guarantee the validity of the theorems we're interested in. In fact, if we dealt with metric spaces only, we could have defined compactness only in terms of sets which satisfy the Bolzano-Weierstrass property. The more general approach we take next, which can be applied in any topological space, uses a different definition which is seemingly unrelated to the Bolzano-Weierstrass property.

Definition 184 A collection $\mathcal{C} = \{A_i : i \in \Lambda, A_i \subset X\}$ **covers** a metric space (X, d) if $X = \cup_{i \in \Lambda} A_i$. \mathcal{C} is called an **open covering** if its elements A_i are open subsets of X .

Definition 185 A metric space (X, d) is **compact** if every open covering \mathcal{C} of X contains a finite subcovering of X .⁷ A subset H of (X, d) is compact if every open covering of H by open sets of X has a finite subcovering of H .

In order to apply this definition to show that a set H is compact we must examine *every* open covering of H and hence it is virtually impossible to use it in determining compactness of a set. The exception is the case of a finite subset H of a metric space X . For if every point $x_n \in H$ is in some open set $A_i \in \mathcal{C}$, then *at most* m carefully selected subsets of \mathcal{C} will have the property that their union contains H . Thus any finite subset $H \subset X$ is compact.

On the other hand, to show that a set H is not compact, it is sufficient to show only *one* open covering cannot be replaced by a finite subcollection that also covers H .

⁷That is, if every open covering \mathcal{C} of X contains a finite subcollection $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$ with $A_{i_j} \in \mathcal{C}$ that also covers X .

Example 186 Let $(X, d) = (\mathbb{R}, |\cdot|)$ and $H = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty)$. Let $\mathcal{C} = \{A_n : n \in \mathbb{N}, A_n = (-1, n)\}$ so that every $A_n \subset \mathbb{R}$ and $H \subset \{\cup_{n \in \mathbb{N}} A_n\}$. If $\{A_{n_1}, A_{n_2}, \dots, A_{n_k}\}$ is a finite subcollection, let $M = \max\{n_1, n_2, \dots, n_k\}$ so $A_{n_j} \subset A_M$ and hence $A_M = \cup_{j=1}^k A_{n_j}$. However, since A_n is open, $M \notin A_M$ and hence the real number $M > 0$ does not belong to a finite open subcovering of H . Thus we have provided one particular covering of H by open sets $(-1, n)$ which cannot be replaced by a finite subcollection that also covers H . This was sufficient to show that H is not compact. This example shows that boundedness of a set is a likely necessary condition for compactness.⁸ See Figure 4.3.1.

Lemma 187 Let $H \subset X$. If H is compact, then H is bounded.

Proof. Let $x_0 \in H$. Let $A_m = \{x \in X : d(x_0, x) < m\}$. Here we construct an increasing nested sequence of open sets A_m whose countable union contains H . That is, $H \subset \cup_{m=1}^{\infty} A_m = X$ and $A_1 \subset A_2 \subset \dots \subset A_m \subset \dots$. It follows from the definition of compactness that there is a finite number M such that $A_1 \subset A_2 \subset \dots \subset A_M$ covers H . Then $H \subset A_M$ and hence bounded. ■

Example 188 $H = [0, 1)$ cannot be covered by a finite subcollection of sets $A_n = (-1, 1 - \frac{1}{n})$ for $n \in \mathbb{N}$. It is simple to see that $H \subset \{\cup_{n \in \mathbb{N}} A_n\}$ and each $A_n \subset \mathbb{R}$. However, if $\{A_{n_1}, A_{n_2}, \dots, A_{n_k}\}$ is a finite subcollection and we let $M = \sup\{n_1, n_2, \dots, n_k\}$, then $A_{n_j} \subset A_M$ and hence $A_M = \cup_{j=1}^k A_{n_j}$. However, since A_n is open, $1 - \frac{1}{M} \notin A_M$ and hence the real number $1 - \frac{1}{M} \in H$ does not belong to a finite open subcovering of H . This example shows that closedness of a set is a likely necessary condition for compactness.⁹ Figure 4.3.2.

Lemma 189 Let $H \subset X$. If H is compact, then H is closed.

Proof. H is closed $\Leftrightarrow X \setminus H$ is open. Let $x \in X \setminus H$ and construct an increasing nested sequence of open sets A_k around but not including x with the property that their countable union is $H \setminus \{x\}$. That is, $A_k = \{y \in X : d(x, y) > \frac{1}{k}, k \in \mathbb{N}\}$ in X . Then $\{H \setminus \{x\}\} = \cup_{k \in \mathbb{N}} A_k$. Since $x \notin H$, each element of H is in some set A_k by an application of Corollary 100 so that $H \subset$

⁸Unboundedness in this example was really just an application of the Archimedean Theorem 99.

⁹Lack of closedness in this example was really just an application of the Corollary to the Archimedean Theorem 100.

$\cup_{k \in \mathbb{N}} A_k$. Since H is compact, it follows from the definition of compactness that there is a finite $K \in \mathbb{N}$ such that $H \subset \cup_{k=1}^K A_k = A_K$. In that case, there is an open ball around x such that $B_{\frac{1}{K+1}}(x) \subset X \setminus H$ with $B_{\frac{1}{K+1}}(x) \cap H = \emptyset$. Since x was arbitrary, each point in the complement of H is contained in an open ball in $X \setminus H$. Thus $X \setminus H$ is open in which case H is closed. See Figure 4.3.3. ■

Theorem 190 *A closed subset of a compact set is compact.*

Proof. Let X be compact and $H \subset X$ be closed. Let $\mathcal{C} = \{A_i\}$ be an open covering of H . Then $\mathcal{G} = \{A_i\} \cup (X \setminus H)$ is an open covering of X since $X \setminus H$ is open (because H is closed). Since X is compact, there exists a finite subcollection \mathcal{F} of \mathcal{G} covering X . Since \mathcal{F} also covers H , then $\mathcal{F} \setminus \{X \setminus H\}$ also covers H and is a subcollection of \mathcal{C} . Then H is compact. ■

Lemmas 187 and 189 provide necessary conditions for a set to be compact. But we would like to have sufficient conditions that guarantee compactness of a set. To that end, Theorem 190 is useful but has limited applicability. The original space has to be compact in order to be able to use it. Are the necessary conditions of Lemmas 187 and 189 in fact sufficient? Not necessarily, as the next example shows.

Example 191 *Consider the metric space (\mathbb{R}, d') with $d'(x, y) = \min\{|x - y|, 1\}$. In this case, \mathbb{R} is bounded since $|x - y| \leq 1, \forall x, y \in \mathbb{R}$. We also know that \mathbb{R} is closed. It is clear, however, that \mathbb{R} is not compact in (\mathbb{R}, d') since a collection $\mathcal{A} = \{(-n, n), n \in \mathbb{N}\}$ covers \mathbb{R} but it doesn't contain a finite subcollection that also covers \mathbb{R} .*

Exercise 4.3.1 *Show that d' is a metric on \mathbb{R} .*

The space $(\mathbb{R}, |\cdot|)$ provides a clue as to a set of sufficient conditions to establish compactness. Since Lemmas 187 and 189 apply to any metric space, we know compactness implies boundedness and completeness. But in $(\mathbb{R}, |\cdot|)$, boundedness and completeness is equivalent to the Bolzano-Weierstrass property by Theorem 121 (which we attributed to Heine-Borel). Thus, isn't the Bolzano-Weierstrass property sufficient for compactness? In $(\mathbb{R}, |\cdot|)$, this is true and we now show that the Bolzano-Weierstrass property is also sufficient in any metric space. Before we do this, we begin by formulating compactness in terms of sequences (consistent with the approach we are taking in this chapter).

Definition 192 A subset H of a metric space X is **sequentially compact** if every sequence in H has a subsequence that converges to a point in H .

Next we turn to establishing that the Bolzano-Weierstrass property, sequential compactness, and compactness are equivalent in any metric space.

Theorem 193 Let (X, d) be a metric space. Let $H \subset X$. The following are equivalent: (i) H is compact; (ii) Every infinite subset of H has a cluster point; (iii) Every sequence in H has a convergent subsequence.

Proof. (Sketch) ($i \Rightarrow ii$) It is sufficient to prove the contrapositive that if $A \subset H$ has no cluster point, then A must be finite. If A has no cluster point, then A contains all its cluster points (because the set of cluster points is empty and every set contains the empty set.). Therefore A is closed. Since A is a closed subset of a compact space H , it is compact. For each $x \in A$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \cap \{A \setminus \{x\}\} = \emptyset$ since x is not a cluster point of A by Definition 140. Thus, the collection $\{B_\varepsilon(x), x \in A\}$ forms an open covering of H . Since H is compact, it is covered by finitely many $B_\varepsilon(x)$. Since each $B_\varepsilon(x)$ contains only one point of H , H is finite.

($ii \Rightarrow iii$) Given $\langle s_i \rangle$, consider the set $S = \{s_i \in H : i \in \mathbb{N}\}$. If S is finite, then $s^* = s_i$ for infinitely many values of i in which case $\langle s_i \rangle$ has a subsequence that is constant (and hence converges automatically). If S is infinite, then by (ii) it has a cluster point s^* . Since s^* is a cluster point, we know by Definition 140 that $\forall \varepsilon = \frac{1}{n}$ there exist $x_{i_n} \in B_{\frac{1}{n}}(s^*)$ such that $s_{i_n} \neq s^*$. This allows us to construct a subsequence $\langle s_{i_1}, s_{i_2}, \dots \rangle$ which converges to s^* .¹⁰

($iii \Rightarrow i$) First, we show that $\forall \varepsilon > 0$, \exists a finite subcovering of H (by ε -balls). Once again, it is sufficient to prove the contrapositive: If for some $\varepsilon > 0$, H has no finite subcover, then H has no convergent subsequence. If H cannot be covered with a finite number of balls, construct $\langle s_i \rangle$ as follows: Choose any $s \in H$, say s_1 . Since $B_\varepsilon(s_1)$ is not all of S (which would contradict that S has no finite subcover), choose $s_2 \notin B_\varepsilon(s_1)$. In general,

¹⁰More specifically, define the subsequence $\langle s_{g(i)} \rangle$ approaching s^* inductively as follows: Choose i_1 such that $s_{i_1} \in B_1(s^*)$. Since s^* is a cluster point of the set X , it is also a cluster point of the set $S_2 = \{s_i \in S : i \in \mathbb{N}, i \geq 2\}$ obtained by deleting a finite number of elements of S . Therefore, there is an element s_{i_2} of S_2 which is an element of $B_{\frac{1}{2}}(s^*)$ with $i_2 > i_1$. Continuing by induction, given i_{n-1} , choose $i_n > i_{n-1}$ such that $s_{i_n} \in B_{\frac{1}{n}}(s^*)$.

given $\langle s_1, s_2, \dots, s_n \rangle$, choose $x_{n+1} \notin B_\varepsilon(s_1) \cup B_\varepsilon(s_2) \cup \dots \cup B_\varepsilon(s_n)$ since these balls don't cover S . By construction $d(s_{n+1}, s_i) \geq \varepsilon$ for $i = 1, \dots, n$. Thus, $\langle s_i \rangle$ can have no convergent subsequence. The above procedure can be used to construct a finite subcollection that covers S .¹¹ ■

When we put this theorem together with the result that every closed and bounded set has the Bolzano-Weierstrass Property we get a simple criterion for determining compactness of a subset in $(\mathbb{R}, |\cdot|)$. In particular, all we have to do is establish that a set in $(\mathbb{R}, |\cdot|)$ is closed and bounded to know it is compact.

Corollary 194 (Heine-Borel) *Given $(\mathbb{R}, |\cdot|)$, $H \subset \mathbb{R}$ is compact iff H is closed and bounded.*

Proof. Follows from Theorem 193 together with the Heine Borel Theorem 121. ■

Corollary 194 is the “more familiar” version of the Heine-Borel Theorem and can easily be extended to \mathbb{R}^n with the Euclidean metric.

Is there any relation between compactness and completeness? While it may not appear so by their definitions, the next result establishes that they are in fact related.

Lemma 195 *Let (X, d) be a metric space. If X is compact, then it is complete.*

Proof. From Theorem 193 every Cauchy sequence has a convergent subsequence. Completeness follows by Lemma 173. ■

The converse of Lemma 195 does not necessarily hold; that is, it doesn't follow that if X is complete, then it is compact as Example 191 shows. We need a stronger condition than boundedness to prove an analogue of the Heine-Borel Corollary 194 for general metric spaces. The condition was actually already used in part (iii) of Theorem 193.

Definition 196 *A metric space (X, d) is **totally bounded** if $\forall \varepsilon > 0$, there is a finite covering of X by ε -balls.*

¹¹See the Lebesgue Number Lemma, p. 179, of Munkres for this construction. For the case of $X = \mathbb{R}^n$, see Bartle Theorem 23.3 p. 160.

As we can see, the definition of total boundedness is quite similar to Definition 185 of compactness. One might ask how to check if a metric space is totally bounded. Though there is no satisfactory answer in a general metric space, there are various criteria for specific spaces. For instance, total boundedness in \mathbb{R}^n is equivalent with boundedness.

Total boundedness of a metric space implies boundedness, but the converse is not true.

Example 197 *While we established that (\mathbb{R}, d') with $d'(x, y) = \min\{|x - y|, 1\}$ was bounded in Example 191, it is not totally bounded. This follows because all of \mathbb{R} cannot be covered with finitely many balls of radius, say $\frac{1}{4}$.*

Next we establish an analogue of the Heine-Borel Theorem for general metric spaces.

Theorem 198 *A metric space (X, d) is compact iff it is complete and totally bounded.*

Proof. (\Rightarrow) Completeness follows by Lemma 195. Total boundedness follows from Definition 196 given X is compact.

(\Leftarrow) By Theorem 193 it suffices to show that if $\langle x_n \rangle$ is a sequence in X , then there exists a subsequence $\langle x_{g(n)} \rangle$ that converges. **Since X is complete**, it suffices to construct a subsequence that is Cauchy. **Since X is totally bounded**, there exist finitely many $\varepsilon = 1$ balls that cover X . At least one of these balls, say B_1 contains y_n for infinitely many indices. Let $J_1 \subset \mathbb{N}$ denote the set of all such indices for which $s_n \in B_1$. Next cover X by finitely many $\varepsilon = \frac{1}{2}$ balls. Since J_1 is infinite, at least one of these balls, say B_2 contains y_n for infinitely many indices. Let $J_2 \subset J_1$ denote the set of all such indices for which $n \in J_1$ and $y_n \in B_2$. Using this construction, we obtain a sequence $\langle J_k \rangle$ such that $J_k \supset J_{k+1}$. If $i, j \geq k$, then $n_i, n_j \in J_k$ and y_{n_i}, x_{n_j} are contained in a ball B_k of radius $\frac{1}{k}$. Hence $\langle x_{n_i} \rangle$ is Cauchy. See Figure 4.3.4. ■

So how does one test for total boundedness? ???EXAMPLE????

4.4 Connectedness

Connectedness of a space is very simple. A space is “disconnected” if it can be broken up into separate globs, otherwise it is connected. More formally,

Definition 199 Let (X, d) be a metric space. $S \subset X$ is **disconnected** (or **separated**) if there exist a pair of open sets T, U such that $S \cap U$ and $S \cap T$ are disjoint, non-empty and have union S . S is **connected** if it is not disconnected. See Figure 4.4.1.

Example 200 (a) Let $(X, d) = (\mathbb{R}, |\cdot|)$ and $H = \mathbb{N}$. Then \mathbb{N} is disconnected in \mathbb{R} since we can take $T = \{x \in \mathbb{R} : x > \frac{3}{2}\}$ and $U = \{x \in \mathbb{R} : x < \frac{3}{2}\}$. Then $T \cap \mathbb{N} \neq \emptyset \neq U \cap \mathbb{N}$, $(T \cap \mathbb{N}) \cap (U \cap \mathbb{N}) = \emptyset$, and $\mathbb{N} = T \cup U$. (b) Let $(X, d) = (\mathbb{R}, |\cdot|)$ and $H = \mathbb{Q}_+$. Then \mathbb{Q}_+ is disconnected in \mathbb{R} since we can take $T = \{x \in \mathbb{R} : x > \sqrt{2}\}$ and $U = \{x \in \mathbb{R} : x < \sqrt{2}\}$. See Figure 4.4.2.

Theorem 201 $I = [0, 1]$ is a connected subset of \mathbb{R} .

Proof. Suppose, to the contrary, that there are two disjoint non-empty open sets A, B whose union is I . Since A and B are open, they do not consist of a single point. WLOG let $a \in A$ and $b \in B$ such that $0 < a < b < 1$. Let $c = \sup\{x \in A : x < b\}$, which exists by Axiom 3. Since $0 < c < 1$, $c \in A \cup B$. If $c \in A$, then $c \neq b$ and since A is open, there is a point $a_1 \in A$ with $c < a_1$ such that $[c, a_1]$ is contained in $\{x \in A : x < b\}$. But this contradicts the definition of c . A similar argument can be made if $c \in B$. ■

This result can easily be extended to any (open, closed, half open, etc.) subset of \mathbb{R} and to show \mathbb{R} itself is connected. Furthermore, it is possible to construct cartesian products of connected sets which are themselves connected.¹²

4.5 Normed Vector Spaces

Before moving onto the next topological concept (continuity), we give an example of a specific type of metric space called a normed vector space. Normed vector spaces are by far the most important type of metric space we will deal with in this book. A normed vector space has features that a metric space doesn't have in general; it possesses a certain algebraic structure. Elements of a vector space (called vectors) can be added, subtracted, and multiplied by a number (called a scalar). See Figure 4.5.1 for the relation between metric spaces and vector spaces.

¹²See Munkres p.150.

Definition 202 A **vector space** (or *linear space*) is a set V of arbitrary elements (called *vectors*) on which two binary operations are defined: (i) closed under vector addition (if $u, v \in V$, then $u + v \in V$) and (ii) closed under scalar multiplication (if $a \in \mathbb{R}$ and $v \in V$, then $av \in V$) which satisfy the following axiom(s):

C1. $u + v = v + u, \forall u, v \in V$

C2. $(u + v) + w = u + (v + w), \forall u, v, w \in V$

C3. $\exists 0 \in V \ni v + 0 = v = 0 + v, \forall v \in V$

C4. For each $v \in V, \exists(-v) \in V \ni v + (-v) = 0 = (-v) + v$

C5. $1v = v, \forall v \in V$

C6. $a(bv) = (ab)v, \forall a, b \in \mathbb{R}$ and $\forall v \in V$

C7. $a(u + v) = au + av, \forall u, v \in V$

C8. $(a + b)v = av + bv, \forall a, b \in \mathbb{R}$ and $\forall v \in V$

Example 203 \mathbb{R} is the simplest vector space. The elements are real numbers where ‘+’ and ‘·’ were introduced in Axiom 1. \mathbb{R}^2 is also a vector space whose basic elements are 2-tuples, say (x_1, x_2) . We interpret (x_1, x_2) not as a point in \mathbb{R}^2 with coordinates (x_1, x_2) but as a displacement from some location. For instance, the vector $(1, 2)$ means move one unit to the right and two units up from your current location. See Figure 4.5.2a for an example of the vector $(1, 2)$ from two different initial locations. Often we take the initial location to be the origin. Vector addition (see Figure 4.5.2b.) is then defined as $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and scalar multiplication (see Figure 4.5.2c.) is defined as $a(x_1, x_2) = (ax_1, ax_2)$. Let $\mathfrak{F}(X, \mathbb{R})$ be the set of all real valued functions $f : X \rightarrow \mathbb{R}$. Then we can define $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$. These two operations satisfy Axioms C1 – C8 and hence $\mathfrak{F}(X, \mathbb{R})$ is a vector space. We will consider such sets extensively in Chapter 6.

Definition 204 A **vector subspace** $U \subset V$ is a subset of V which is a vector space itself.

Example 205 Let $V = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ and $U = \{(x, y) : y = 2x, \text{ where } x, y \in \mathbb{R}\}$. Then U is a vector subspace of V . Note that $Z = \{(x, y) : y = 2x + 1, \text{ where } x, y \in \mathbb{R}\}$ is a subset of V but it is not a vector subspace of V since $0 \notin Z$.

The algebraic structure of a vector space by itself doesn't allow us to measure distance between elements and hence doesn't allow us to define topological properties. This can be accomplished in a vector space through a distance function called the norm.

Definition 206 If V is a vector space, then a **norm** on V is a function from V to \mathbb{R} , denoted $\|\cdot\| : V \rightarrow \mathbb{R}$, which satisfies the following properties $\forall u, v \in V$ and $\forall a \in \mathbb{R} : (i) \|v\| \geq 0$, $(ii) \|v\| = 0$ iff $v = 0$, $(iii) \|av\| = |a| \|v\|$, and $(iv) \|u + v\| \leq \|u\| + \|v\|$. A vector space in which a norm has been defined is called a **normed space**.

Notice that the algebraic operations vector addition and scalar multiplication are used in defining a norm. Thus, a norm cannot be defined in a general metric space which is not equipped with these operations. But a vector space equipped with a norm can be seen as a metric space and a metric space which has a linear structure is also a normed vector space. The following theorem establishes this relationship.

Theorem 207 Let V be a vector space then

(i) If (V, d) is a metric space then $(V, \|\cdot\|)$ is a normed vector space with the norm $\|\cdot\| : V \rightarrow \mathbb{R}$ defined $\|x\| = d(x, 0)$, $\forall x \in V$

(ii) If $(V, \|\cdot\|)$ is a normed vector space then (V, ρ) is a metric space with the metric $\rho : V \times V \rightarrow \mathbb{R}$ defined $\rho(x, y) = \|x - y\|$, $\forall x, y \in V$

Exercise 4.5.1 Prove Theorem 207.

Note that whenever a metric space has the additional algebraic structure given in 202, we will use the norm rather than the metric and hence work in normed vector spaces.

Exercise 4.5.2 Redefine convergence, open balls, and boundedness in terms of normed vector spaces.

Definition 208 A complete normed vector space is called a **Banach space**.

Some vector spaces are endowed with another operation, called an inner (or dot) product that assigns a real number to each pair of vectors. The inner product enables us to measure the “angle” between elements of a vector space.¹³

Definition 209 *If V is a vector space, then an **inner product** is a function $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$ which satisfies the following properties $\forall u, v, w \in V$ and $\forall a \in \mathbb{R}$: (i) $\langle v, v \rangle \geq 0$, (ii) $\langle v, v \rangle = 0$ iff $v = 0$, (iii) $\langle u, v \rangle = \langle v, u \rangle$, (iv) $\langle u, (v + w) \rangle = \langle u, v \rangle + \langle u, w \rangle$, (v) $\langle (au), v \rangle = a \langle u, v \rangle = \langle u, (av) \rangle$. A vector space in which an inner product has been defined is called an **inner product space**.*

The inner product can be used to define a norm (in particular the Euclidean measure of distance) in the following way.

Theorem 210 *Let V be an inner product space and define $\|v\| = \sqrt{\langle v, v \rangle}$. Then $\|\cdot\|: V \rightarrow \mathbb{R}$ is a norm which satisfies the Cauchy-Schwartz inequality $\langle u, v \rangle \leq \|u\|\|v\|$.*

Proof. (Sketch) Since $\langle v, v \rangle \geq 0$ by part (i) of definition 209, $\sqrt{\langle v, v \rangle}$ exists and exceeds zero, establishing part (i) of definition 206. Part (ii) also follows from (ii) of definition 209. By part (v) of definition 209, $\sqrt{\langle (av), (av) \rangle} = \sqrt{a^2 \langle v, v \rangle} = |a| \sqrt{\langle v, v \rangle} = |a| \|v\|$, establishing part (iii). To establish Cauchy-Schwartz, let $w = au - bv$ for $a, b \in \mathbb{R}$ and $u, v \in V$. By definition 209, $w \in V$. Then

$$\begin{aligned} 0 &\leq \langle w, w \rangle = a^2 \langle u, u \rangle - 2ab \langle u, v \rangle + b^2 \langle v, v \rangle \\ &= \|v\|^2 \|u\|^2 - 2\|v\|\|u\| \langle u, v \rangle + \|u\|^2 \|v\|^2 \\ &= 2\|u\|\|v\| (\|u\|\|v\| - \langle u, v \rangle) \end{aligned}$$

where the second equality follows by letting $a = \|v\|$ and $b = \|u\|$, which were free parameters in the first place. ■

To get some intuition for this result, notice that if θ is the angle between vectors u and v , then the relationship between the inner product and norms of the vectors is given by $\langle u, v \rangle = \|v\|\|u\| \cos \theta$. The inequality then follows since $\cos \theta \in [-1, 1]$. See Figure 4.5.6 for this geometric interpretation of the Cauchy Schwartz inequality.

¹³For instance, orthogonality is just $\langle u, v \rangle = 0$.

Exercise 4.5.3 Finish the proof of Theorem 210 (i.e. establish the triangle inequality in part (iv) of definition 206).

Whereas some norms (eg. the Euclidean norm) can be induced from an inner product, other norms (eg sup norm) cannot be.

Definition 211 A complete inner product space is called a **Hilbert space**.

Note that a Hilbert space is also a Banach space.

4.5.1 Convex sets

Definition 212 We say that a **linear combination** of $x_1, \dots, x_n \in V$ is $\{\sum_{i=1}^n \alpha_i x_i, \alpha_i \in \mathbb{R}, i = 1, \dots, n\}$. We say that a **convex combination** of $x_1, \dots, x_n \in V$ is $\{\sum_{i=1}^n \alpha_i x_i, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1, i = 1, \dots, n\}$.

Definition 213 A subset S of a vector space V is a **convex set** if for every $x, y \in S$, the convex combination $\alpha x + (1 - \alpha)y \in S$, for $0 \leq \alpha \leq 1$.¹⁴

Example 214 In \mathbb{R} , any interval (e.g. (a, b)) is convex but $(a, b) \cup (c, d)$ with $b < c$ is not convex. See Figure 4.5.3 for convex sets in \mathbb{R}^2 .

Definition 215 The **sum (difference)** of two subsets S_1 and S_2 of a vector space V is $S_1 \pm S_2 = \{v \in V : v = x \pm y, x \in S_1, y \in S_2\}$. See Figure 4.5.4.

Theorem 216 (Properties of Convex Sets) If K_1 and K_2 are convex sets, then the following sets are convex: (i) $K_1 \cap K_2$; (ii) λK_1 ; (iii) $K_1 \pm K_2$.

Proof. (iii) Let $x, y \in K_1 + K_2$ so that $x = x_1 + x_2, x_1 \in K_1, x_2 \in K_2, y = y_1 + y_2, y_1 \in K_1, y_2 \in K_2$. Then $\alpha x + (1 - \alpha)y = \alpha(x_1 + x_2) + (1 - \alpha)(y_1 + y_2) = (\alpha x_1 + (1 - \alpha)y_1) + (\alpha x_2 + (1 - \alpha)y_2) \equiv z_1 + z_2$. Since K_1 and K_2 are convex, $z_1 \in K_1, z_2 \in K_2$. Thus, $x + y \in K_1 + K_2$. ■

Example 217 It is simple to show cases where K_1 and K_2 are convex sets, but $K_1 \cup K_2$ is not convex. See Example 214 in \mathbb{R} .

¹⁴For instance, for $x_1, \dots, x_n \in S$, the convex combination is $\sum_{i=1}^n \alpha_i x_i$, where $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$.

As we will see later, convexity of a set is a desirable property. If a set S is not convex, we may replace it with the smallest convex set containing S called the convex hull.

Definition 218 Let $S \subset V$. The **convex hull** of S is the set of all convex combinations of elements from S , denoted $\text{co}(S)$. That is, $\text{co}(S) = \{x \in V : x = \sum_{i=1}^n \alpha_i x_i, x_i \in S, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1\}$.

Example 219 In \mathbb{R}^n , consider two vectors $A \neq B$. Then $\text{co}(\{A, B\})$ is just a line segment with endpoints A, B . If A, B, C do not lie on the same line, then $\text{co}(\{A, B, C\})$ is the triangle A, B, C . See Figure 4.5.5.

Exercise 4.5.4 Show that if V is convex, then (i) $\text{co}(V) = V$, and (ii) if $S \subset V$, then $\text{co}(S) \subset V$ is the smallest convex set containing S .

4.5.2 A finite dimensional vector space: \mathbb{R}^n

The most familiar vector space is just \mathbb{R}^n , with $n \in \mathbb{N}$ and $n < \infty$. \mathbb{R}^n is the collection of all ordered n -tuples (x_1, x_2, \dots, x_n) with $x_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. Vector addition is defined as $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ and scalar multiplication is defined as $a(x_1, x_2, \dots, x_n) = (ax_1, ax_2, \dots, ax_n)$.

Exercise 4.5.5 Verify \mathbb{R}^n is a vector space under these operations.

Example 220 Since $(\mathbb{R}, |\cdot|)$ is a complete metric space with absolute value metric, it is a Banach space with the norm $\|x\| = |x|$. Since $(\mathbb{R}^n, \|x\|)$ is a complete metric space with Euclidean metric, it is a Banach space with Euclidean norm $\|x\|_2 = \sqrt{\sum_{i=1}^n (x_i)^2}$. Since $(\mathbb{R}^n, \|x\|_\infty)$ is a complete metric space with supremum metric, it is also a Banach space with sup norm $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$.

The next result provides a useful characterization of the relationship between $\|x\|_\infty$ and $\|x\|$.

Theorem 221 If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$.

Proof. Since $(\|x\|_2)^2 = \sum_{i=1}^n (x_i)^2$, it is clear that $|x_i| \leq \|x\|_2, \forall i$. Similarly, if $M = \max\{|x_1|, \dots, |x_n|\}$, then $(\|x\|_2)^2 \leq nM^2$, so $\|x\| \leq \sqrt{n}M$. ■

Example 220 shows that \mathbb{R}^n can be endowed with two different norms. One might ask if these two normed vector spaces are somehow related and if so, in what sense? Distances between two points with respect to these two norms are generally different. See Figure 4.5.7 In this case, these two normed vector spaces are *not* isometric. On the other hand these two spaces have identical topological properties like openness, closeness, compactness, connectedness, and continuity. In this case, we say that these two normed vector spaces are **homeomorphic** or topologically equivalent.

To show that two metric spaces (or normed vector spaces according to Theorem 207) are topologically equivalent it suffices to show that the collections of open sets in both spaces are identical. This is because all topological properties can be defined in terms of open sets. The fact that open sets are identical follows Theorem 221. To see this, let A be open in \mathbb{R}^n under Euclidean norm. Then $\forall x \in A, \exists \varepsilon > 0$ such that $\{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\} \subset A$. But from the first part of the inequality in Theorem 221 we know $\{y \in \mathbb{R}^n : \|y - x\|_\infty \leq \|y - x\| < \varepsilon\} \subset A$. Hence A is open in \mathbb{R}^n under the sup norm. The inverse can be shown the same way using the second part of the inequality. We will discuss this at further length after we introduce continuity.

Example 222 In \mathbb{R}^n , define $\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$. \mathbb{R}^n with inner product defined this way is a Hilbert space.

Exercise 4.5.6 Verify the dot product in Example 222 defines an inner product on \mathbb{R}^n .

Theorem 223 In the Euclidean space \mathbb{R}^n a sequence of vectors $\langle x_m \rangle$ converges to a vector $x = (x_1, \dots, x_n)$ if and only if each component $\langle x_m^i \rangle$ converges to $x^i, i = 1, \dots, n$.

Exercise 4.5.7 Prove Theorem 223.

We next introduce the simplest kind of convex set in \mathbb{R}^n .

Definition 224 A nondegenerate **simplex** in \mathbb{R}^n is the set of all points

$$S = \{x \in \mathbb{R}^n : x = \alpha_0v^0 + \alpha_1v^1 + \dots + \alpha_nv^n, \alpha_0 \geq 0, \dots, \alpha_n \geq 0 \text{ and } \sum_{i=0}^n \alpha_i = 1\} \quad (4.2)$$

where v^0, v^1, \dots, v^n are vectors from \mathbb{R}^n such that $v^1 - v^0, v^2 - v^0, \dots, v^n - v^0$ are linearly independent. Vectors v^0, v^1, \dots, v^n are called *vertices*. The numbers $\alpha_0, \dots, \alpha_n$ are called **barycentric coordinates** (the weights of the convex combinations with respect to $n + 1$ fixed vertices) of the point x .

Example 225 A nondegenerate simplex in \mathbb{R}^1 is a line segment, in \mathbb{R}^2 is a triangle, in \mathbb{R}^3 is a tetrahedron. In \mathbb{R}^3 , for example, the simplex is determined by 4 vertices, any 3 vertices determine a **boundary face**, any 2 vertices determine a **boundary segment**. See Figure 4.5???(4.8.2.???)

A simplex is just the convex hull of the set of all vertices $V = \{v^0, v^1, \dots, v^n\}$. By the following theorem, any point of a convex hull of V can be expressed as a convex combination of these vertices. Do not confuse the $n + 1$ barycentric coordinates (the α_i) of x with the n cartesian coordinates of x .

Theorem 226 (Caratheodory) If $X \subset \mathbb{R}^n$ and $x \in \text{co}(X)$, then $x = \sum_{i=1}^{n+1} \lambda_i x_i$

for some $\lambda_i \geq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, $x_i \in X$, $\forall i$.

Proof. (Sketch) Since $x \in \text{co}(X)$, it can be written as a convex combination of m points by Theorem ???. If $m \leq n + 1$, we are done. If not, then the generated vectors

$$\begin{bmatrix} x_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} x_m \\ 1 \end{bmatrix}$$

are linearly dependent, so a combination of them will be zero (i.e.

$$\sum_{i=1}^m \mu_i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = 0$$

with μ_i not all zero. If λ_i are coefficients of x_i , we can choose α to reduce the number of vectors with nonzero coefficients below m by setting $\theta_i \equiv \lambda_i - \alpha \mu_i$.

■

We know that in \mathbb{R}^n each vector can be written as a linear combination of n -linearly independent vectors (called a basis). That is, $x = \sum_{i=1}^n \alpha_i x_i$,

$\{x_1, \dots, x_n\}$ is a basis. There is no restriction on the coefficients α_i . In Theorem 226 there are additional assumptions put on α_i (i.e. $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$). Now adding one more variable (from n to $n + 1$) to the system yields a unique solution for vectors belonging to the $co(V)$ and no solution for other vectors.

The following two examples demonstrate the difference between cartesian coordinates and barycentric coordinates in \mathbb{R}^2 .

Example 227 Let $V = \{(0, 1), (1, 0), (1, 1)\}$. Say we want to express the vector $(\frac{2}{3}, \frac{2}{3})$ as a linear combination of $(0, 1)$ and $(1, 0)$, two basis vectors. That is $(\frac{2}{3}, \frac{2}{3}) = \frac{2}{3}(1, 0) + \frac{2}{3}(0, 1)$, but this is not a convex combination since $\frac{2}{3} + \frac{2}{3} = \frac{4}{3} \neq 1$. But any point from $co(S)$ can be uniquely expressed as the convex combination of vectors from S . For instance,

$$\left(\frac{2}{3}, \frac{2}{3}\right) = \alpha(1, 0) + \beta(0, 1) + (1 - \alpha - \beta)(1, 1)$$

where $0 \leq \alpha, \beta \leq 1$. Letting $\alpha = \beta = \frac{1}{3}$, we have

$$\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3}(1, 0) + \frac{1}{3}(0, 1) + \frac{1}{3}(1, 1)$$

On the other hand, a vector outside $co(V)$ (like $(\frac{1}{3}, 0)$) cannot be expressed as a convex combination of vectors from V . See Figure 4.5.8.

Example 228 Let the fixed vertices be given by $v^0 = (0, 1)$, $v^1 = (0, 3)$, $v^2 = (2, 0)$ and consider the point $x^1 = (1, 1)$ on the interior of the simplex. See Figure 4.5.8. (4.8.3.) The barycentric coordinates of x^1 with respect to vertices v^0 , v^1 , v^2 are $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ since $(1, 1) = \frac{1}{4}(0, 1) + \frac{1}{4}(0, 3) + \frac{1}{2}(2, 0)$. In the case of $x^2 = (0, 2)$, the barycentric coordinates are $(\frac{1}{2}, \frac{1}{2}, 0)$ since $(0, 2) = \frac{1}{2}(0, 1) + \frac{1}{2}(0, 3) + 0(2, 0)$. In the case of $x^3 = (2, 0)$, the barycentric coordinates are $(0, 0, 1)$ since $(2, 0) = 0 \cdot (0, 1) + 0 \cdot (0, 3) + 1 \cdot (2, 0)$. Notice that x^1 is an interior point of the simplex so that all its barycentric coordinates are positive, that x^2 is on the boundary so that one barycentric coordinate is 0, and x^3 is a vertex so that it has 2 barycentric coordinates which are zeros. In this section, we will always mean by α_i barycentric coordinates of a point inside the fixed simplex (including boundary points).

The next result, while purely combinatorial, will be used in the proof of Brouwer's Fixed Point Theorem 302. While this can be proven for \mathbb{R}^n , here we present it for \mathbb{R}^2 . First we must introduce an indexing scheme for points in the simplex as follows. Let Z be a set of labels in \mathbb{R}^2 given by $\{0, 1, 2\}$. While the index function $I : S \rightarrow Z$ can obtain any value from Z for points x inside the simplex, it must satisfy the following restrictions on the boundary:

$$I(x) = \begin{cases} 0 \text{ or } 1 & \text{on the line segment } \overline{(v^0, v^1)} \\ 0 \text{ or } 2 & \text{on the line segment } \overline{(v^0, v^2)} \\ 1 \text{ or } 2 & \text{on the line segment } \overline{(v^1, v^2)} \end{cases} \quad (4.3)$$

For example on the boundary $\overline{(v^0, v^1)}$, $I(x)$ can't obtain the value 2. Thus $I(x) = 0$ or 1 on the line segment $\overline{(v^0, v^1)}$. See Figure 4.5(4.8.6). Note that the (4.3) implies that $I(v^0) = 0$, $I(v^1) = 1$, and $I(v^2) = 2$ at the vertices.

Lemma 229 (Sperner) *Form the barycentric subdivision of a nondegenerate simplex. Label each vertex with an index $I(x) = 0, 1, 2$ that satisfies the restrictions (4.3) on the boundary. Then there is an odd number of cells (thus at least 1) in the subdivision that have vertices with the complete set of labels 0, 1, 2.*

Proof. By induction on n . We show just the first step (i.e. for $n = 1$) to get the idea. If $n = 1$, a nondegenerate simplex is a line segment and a face is a point. To obey the restrictions (4.3), one end has label 0 the other has label 1, and the rest is arbitrary. See Figure 4.5(4.8.11). Next define a counting function F , where by $F(a, b)$ we mean the number of elements in the simplex of type (a, b) . For example, in Figure 4.5(4.8.11), $F(0, 0) = 2$, $F(0, 1) = 3$, $F(1, 1) = 1$, $F(0) = 4$, $F(1) = 3$. Permutations don't matter (i.e. $(0, 1)$ and $(1, 0)$ are the same type which is why $F(0, 1) = 3$ since we have two occurrences of $\overline{(0, 1)}$ and one of $\overline{(1, 0)}$. Consider the single points labeled 0. Two labels 0 occur in each cell of type $(0, 0)$, one label 0 occurs in each cell of type $(0, 1)$. The sum $2F(0, 0) + F(0, 1)$ counts every interior 0 twice, since every interior 0 is the point that is shared by two cells and the sum counts every boundary 0 once. Therefore $2F(0, 0) + F(0, 1) = 2F_i(0) + F_b(0)$ where $F_i(0)$ is the number of i (for interior) 0's and $F_b(0)$ is the number of b (for boundary) 0's. Clearly $F_b(0) = 1$. Hence

$$F(0, 1) = 2[F_i(0) - F(0, 0)] + 1. \quad (4.4)$$

In Figure 4.5(4.8.9????) these numbers are $F(0, 1) = 3$, $F_i(0) = 3$, $F(0, 0) = 2$. In 1 dimension the number of cells having vertices with the complete set of labels 0, 1 is $F(0, 1)$ and from (4.4) we see that it is always an odd number.

■

Example 230 *The values of counting functions F for the simplex in Figure 4.5(4.8.9????) are:*

$$\begin{aligned} F(0) &= 7, F(0, 0) = 4, F(1, 1) = 8, F(0, 0, 0) = 0 \\ F(1) &= 9, F(0, 1) = 16, F(1, 2) = 8, F(0, 0, 1) = 5 \\ F(2) &= 5, F(0, 2) = 8, F(2, 2) = 1, F(0, 0, 2) = 2 \\ F(0, 1, 1) &= 6, F(0, 2, 2) = 1, F(1, 1, 1) = 2, F(2, 2, 2) = 0 \\ F(0, 1, 2) &= 7, F(1, 1, 1) = 2, F(1, 2, 2) = 0 \end{aligned}$$

4.5.3 Series

The fact that a normed vector space is the synthesis of two structures - topological and algebraic - enables us to introduce the notion of an infinite sum (i.e. a sum containing infinitely many terms). These objects are called *series*. As we will see in the subsection on ℓ_p spaces, norms will be defined in terms of functions of infinite sums so understanding when they converge or diverge is critical.

Let $(V, \|\cdot\|)$ be a normed vector space and let $\langle x_n \rangle$ be a sequence in V .

We can define a new sequence $\langle y_n \rangle$ by $y_n = \sum_{i=1}^n x_i$. The sequence $\langle y_n \rangle$ is called the *sequence of partial sums* of $\langle x_n \rangle$. Since X is also a metric space, we can ask if $\langle y_n \rangle$ is convergent (i.e. if there exists an element $y \in X$ such that $\langle y_n \rangle \rightarrow y$ or equivalently $\|y_n - y\|_X \rightarrow 0$). If such an element exists we say that the series $\sum_{i=1}^{\infty} x_i$ is convergent and write $y = \sum_{i=1}^{\infty} x_i$. If $\langle y_n \rangle$ is

not convergent, we say that $\sum_{i=1}^{\infty} x_i$ is divergent.

Example 231 *Consider $(\mathbb{R}, |\cdot|)$ and let $\langle x_n \rangle = \langle \frac{1}{2^n} \rangle$, which is just a geometric sequence with quotient $\frac{1}{2}$. The sequence of partial sums is $y_1 = \frac{1}{2}$,*

$$y_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, y_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \dots, y_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = \frac{1}{2} \left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \right) = 1 - \frac{1}{2^n}.$$

Since $\langle y_n \rangle = \langle 1 - \frac{1}{2^n} \rangle \rightarrow 1$, we write $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$.

Example 232 While we have already seen that $\langle \frac{1}{n} \rangle$ converges (to 0), the harmonic series $\sum_{i=1}^{\infty} \frac{1}{n}$ diverges (i.e. is not bounded). To see this, note

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \end{aligned}$$

The right hand side is the sum of infinitely many halves which is not bounded.

Elements of a series can also be functions. We will deal with series of functions in Chapter 6.

4.5.4 An infinite dimensional vector space: ℓ_p

The example in the above subsection is of a finite dimensional vector space; that is, the Euclidean space \mathbb{R}^n with either norm (there are at most n linearly independent vectors in \mathbb{R}^n). Now we introduce an infinite dimensional vector space. As you will see, results from finite dimensional vector spaces cannot be generalized in infinite dimensional vector spaces.

Definition 233 Let \mathbb{R}^{ω} be the set of all sequences in \mathbb{R} . Let $1 \leq p \leq \infty$ and let ℓ_p be the subset of \mathbb{R}^{ω} whose elements satisfy the $\sum_{i=1}^{\infty} |x_i|^p < \infty$.¹⁵ The ℓ_p -**norm** of a vector $x \in \ell_p$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

and ℓ_{∞} is the subset of all bounded sequences equipped with the norm

$$\|x\|_{\infty} = \sup\{|x_1|, \dots, |x_n|, \dots\}.$$

¹⁵Recall, $\mathbb{R}^{\omega} = \{f : \mathbb{N} \rightarrow \mathbb{R}\}$ where $\omega = \text{card}(\mathbb{N})$.

We note that there is a set of infinitely many linearly independent vectors in ℓ_p , namely $\{e_i = \langle x_j \rangle, i \in \mathbb{N} \text{ where } x_j = 0 \text{ for } i \neq j \text{ and } x_j = 1 \text{ for } i = j\}$ which is called a **basis**.

Before proving that ℓ_p is a Banach space, we use the following example to illustrate some differences between finite dimensional Euclidean space \mathbb{R}^n and infinite dimensional ℓ_2 . In particular, convergence by components is not sufficient for convergence in ℓ_2 (i.e. the result of Theorem 223 is not necessarily true).

Example 234 Let $K = \{e_i = \langle x_j \rangle, i \in \mathbb{N} \text{ where } x_j = 0 \text{ for } i \neq j \text{ and } x_j = 1 \text{ for } i = j\}$. That is,

$$\begin{array}{rcccccccc} e_1 & = & \langle & 1, & 0, & 0, & 0, & \dots & \rangle \\ e_2 & = & \langle & 0, & 1, & 0, & 0, & \dots & \rangle \\ e_3 & = & \langle & 0, & 0, & 1, & 0, & \dots & \rangle \\ e_4 & = & \langle & 0, & 0, & 0, & 1, & \dots & \rangle \\ & & \cdot & & \cdot & & \cdot & & \cdot \\ & & \cdot & & \cdot & & \cdot & & \cdot \\ & & & & \downarrow & \downarrow & \downarrow & \downarrow & \\ & & & & 0 & 0 & 0 & 0 & \end{array}$$

Observe that each component $\langle x_n^i \rangle$ converges to 0 in $(\mathbb{R}, |\cdot|)$ for each $i \in \mathbb{N}$. But the sequence $\langle e_i \rangle$ doesn't converge to $\mathbf{0}$ since $\|e_i - \mathbf{0}\|_2 = 1, \forall i \in \mathbb{N}$. In fact $\langle e_i \rangle$ has no convergent subsequence since the distance between any two elements e_i and $e_j, i \neq j$, is $\|e_i - e_j\|_2 = \sqrt{2}$. Thus according to Theorem 193, K is not compact in ℓ_2 . But notice that K is both bounded and closed and these two properties are sufficient for compactness in \mathbb{R}^n . Notice that K is not totally bounded. For if $\varepsilon = \frac{1}{2}$, the only non-empty subsets of K with diameter less than ε are the singleton sets with one point. Accordingly, the infinite subset K cannot be covered by a finite number of disjoint subsets each with diameter less than $\frac{1}{2}$.

Now we prove that the ℓ_p space is a complete normed vector space (and hence that it is a Banach space for any p satisfying $1 \leq p \leq \infty$) and that ℓ_2 is a Hilbert space with the inner product defined by $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$. First, we need to show that $\|\cdot\|_p$ defines a norm. On $\ell_p, 1 \leq p \leq \infty$. The important role in investigating ℓ_p plays another space ℓ_q whose exponent q is associated with p by the relation $\frac{1}{p} + \frac{1}{q} = 1$ where p, q are non-negative

extended real numbers. Two such numbers are called (mutually) conjugate numbers. If $p = 1$ the conjugate is $q = \infty$ since $\frac{1}{1} + \frac{1}{\infty} = 1 + 0 = 1$. Also notice that $q = \frac{p}{p-1} > 1$ for $p > 1$. If $p = 2$, then $q = 2$. It is straightforward to show that $\|\cdot\|_p$ satisfies the first three properties of a norm. The triangle property is a tricky one. Before showing it we shall establish some important inequalities.

Lemma 235 *Let $a, b > 0$ and $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, with equality if $a^p = b^q$.*

Proof. Since the exponential function is convex, we have $\exp(\lambda A + (1 - \lambda) B) \leq \lambda \exp A + (1 - \lambda) \exp B$, for any real numbers A and B . By substituting $A = p \log a$, $\lambda = \frac{1}{p}$, $B = q \log b$, and $1 - \lambda = \frac{1}{q}$, we get the desired inequality. See Figure 4.5.9. ■

The next result is the analogue of Cauchy-Schwartz in infinite dimensions.

Theorem 236 (Hölder inequality) *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. If the sequences $\langle x_n \rangle \in \ell_p$ and $\langle y_n \rangle \in \ell_q$, then $\langle x_n y_n \rangle \in \ell_1$ and*

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|\langle x_n \rangle\|_p \|\langle y_n \rangle\|_q \quad (= \|x\|_p \|y\|_q) \quad (4.5)$$

where $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$

Proof. For $p = 1$, $q = \infty$, we have

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left\{ \sup \{y_n, n \in N\} \cdot \sum_{n=1}^{\infty} |x_n| \right\} = \|\langle x_n \rangle\|_1 \|\langle y_n \rangle\|_{\infty}.$$

Next, let $p, q \in (1, \infty)$. If $\langle x_n \rangle$ or $\langle y_n \rangle$ is a zero vector, we have equality in (4.5). Now let $\langle x_n \rangle \neq 0, \langle y_n \rangle \neq 0$.¹⁶ Substituting $x_n = \frac{|\langle x_n \rangle|}{\|x\|_p}, y_n = \frac{|\langle y_n \rangle|}{\|y\|_q}$ for ab in lemma 235, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|\langle x_n \rangle|}{\|x\|_p} \cdot \frac{|\langle y_n \rangle|}{\|y\|_q} &\leq \frac{1}{p} \sum_{n=1}^{\infty} \left(\frac{|\langle x_n \rangle|}{\|x\|_p} \right)^p + \frac{1}{q} \sum_{n=1}^{\infty} \left(\frac{|\langle y_n \rangle|}{\|y\|_q} \right)^q \\ &\leq \frac{1}{p} \frac{1}{\left(\frac{1}{\|x\|_p} \right)^p} \cdot \left(\frac{1}{\|x\|_p} \right)^p + \frac{1}{q} \frac{1}{\left(\frac{1}{\|y\|_q} \right)^q} \cdot \left(\frac{1}{\|y\|_q} \right)^q \\ &\leq \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

¹⁶Note this means that not all terms in the sequence equal 0 (i.e. there is at least one term different from 0).

By multiplying $\|x\|_p \cdot \|y\|_q$ we get the result. ■

Note that if $p = q = 2$, Inequality (4.5) is called the *Cauchy-Schwartz inequality*.

Now we can prove that the $\|\cdot\|_p$ norm satisfies the triangle inequality.

Theorem 237 (Minkowski) *Let $1 \leq p \leq \infty$, $x = \langle x_n \rangle$, $y = \langle y_n \rangle \in \ell_p$. Then*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p. \quad (4.6)$$

Proof. If $p = 1$ or $p = \infty$, the proof is trivial. Let $p \in (1, \infty)$. By multiplying both sides of (4.6) by $\left(\|x + y\|_p\right)^{p-1}$ we get the equivalent inequality $\left(\|x + y\|_p\right)^p \leq \left(\|x\|_p + \|y\|_p\right) \left(\|x + y\|_p\right)^{p-1}$. A simple calculation shows this is equivalent to $\sum_{i=1}^{\infty} |x_i| (|x_i + y_i|)^{p-1} + \sum_{i=1}^{\infty} |y_i| (|x_i + y_i|)^{p-1} \leq \|x\|_p \cdot \left(\|x + y\|_p\right)^{p-1} + \|y\|_p \cdot \left(\|x + y\|_p\right)^{p-1}$. Due to symmetry of x, y , it now suffices to show that

$$\sum_{i=1}^{\infty} |x_i| (|x_i + y_i|)^{p-1} \leq \|x\|_p \cdot \left(\|x + y\|_p\right)^{p-1}. \quad (4.7)$$

Let $z_i = (|x_i + y_i|)^{p-1}$ then $\|z\|_q = \left(\sum_{i=1}^{\infty} (z_i)^q\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{\infty} (|x_i + y_i|)^{(p-1)q}\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{\frac{p-1}{p}} = \|x + y\|_p^{p-1}$ where we used the fact that $q(p-1) = p$ and $\frac{1}{q} = \frac{p-1}{p}$. Now by Hölder inequality (4.5), we have $\sum_{i=1}^{\infty} |x_i \cdot z_i| \leq \|x\|_p \cdot \|z\|_q$ which by plugging in z_i yields $\sum_{i=1}^{\infty} |x_i| (|x_i + y_i|)^{p-1} \leq \|x\|_p \cdot \left(\|x + y\|_p\right)^{p-1}$ is just inequality (4.7). ■

Now that we showed that for $1 \leq p \leq \infty$, ℓ_p with $\|\cdot\|_p$ is a normed vector space, we ask "Is it complete?" The answer is yes as the following theorem shows.

Theorem 238 *For $1 \leq p \leq \infty$, the ℓ_p space is a complete normed vector space (i.e. a Banach space).*

Proof. First we show it for $1 \leq p < \infty$. Let $\langle x_m \rangle$ be a Cauchy sequence in ℓ_p , where $x_m = \langle \xi_i^{(m)} \rangle$ (Note that $\langle x_m \rangle$ is a sequence of sequences) such

that $\sum_{i=1}^{\infty} \left| \xi_i^{(m)} \right|^p < \infty$ ($m = 1, 2, \dots$). Since $\langle x_m \rangle$ is Cauchy with respect to $\|\cdot\|_p$, this means that for $\varepsilon \in (0, 1)$, $\exists N$ such that

$$\begin{aligned} \|x_m - x_n\|_p &= \left(\sum_{i=1}^{\infty} \left| \xi_i^{(m)} - \xi_i^{(n)} \right|^p \right)^{\frac{1}{p}} < \varepsilon, \forall m, n \geq N \quad (4.8) \\ \implies \left| \xi_i^{(m)} - \xi_i^{(n)} \right| &< \varepsilon, \forall m, n \geq N, i = 1, 2. \end{aligned}$$

This shows that for each fixed i the sequence $\left\langle \xi_i^{(m)} \right\rangle_{m=1}^{\infty}$ (i th component of $\langle x_n \rangle$) is a Cauchy sequence in \mathbb{R} . Since $(\mathbb{R}, |\cdot|)$ is complete, it converges in \mathbb{R} . Let $\xi_i^{(m)} \rightarrow \xi_i^*$ as $m \rightarrow \infty$ which generates a sequence $x = \langle \xi_1^*, \xi_2^*, \dots \rangle$. We must show that $x \in \ell_p$ and $x_n \rightarrow x$ with respect to the $\|\cdot\|_p$ norm. From (4.8) we have $\sum_{i=1}^k \left| \xi_i^{(m)} - \xi_i^{(n)} \right|^p < \varepsilon^p$ for $m, n \geq N, k \in \mathbb{N}$. Letting $n \rightarrow \infty$ we obtain $\sum_{i=1}^k \left| \xi_i^{(m)} - \xi_i \right|^p \leq \varepsilon^p, \forall m \geq N, k \in \mathbb{N}$ and letting $k \rightarrow \infty$ gives

$$\sum_{i=1}^{\infty} \left| \xi_i^{(m)} - \xi_i^* \right|^p \leq \varepsilon^p \leq \varepsilon, \forall m \geq N. \quad (4.9)$$

This shows that $x_m - x = \left\langle \xi_i^{(m)} - \xi_i^* \right\rangle \in \ell_p$. Since $x_m \in \ell_p$, it follows by the Minkowski Theorem 237 that $\|x\|_p = \|x_m + (x - x_m)\|_p \leq \|x_m\|_p + \|(x - x_m)\|_p$ for $x \in \ell_p$. Furthermore, if $p = \infty$, from (4.9) we obtain $\|x_m - x\|_p < \varepsilon, \forall m \geq N$ which means $x_m \rightarrow x$ with respect to the $\|\cdot\|_p$ norm. ■

The proof works by taking a Cauchy sequence in ℓ_p (say $\langle x_1 \rangle, \langle x_2 \rangle, \dots, \langle x_m \rangle, \dots$) and showing that a sequence of components (say the first one is $\langle \xi_1^1, \xi_1^2, \dots, \xi_1^m, \dots \rangle$) is also Cauchy in \mathbb{R} (converging to say ξ_1^*). Then we show the original sequence of sequences converges to the sequence $\langle \xi_1^*, \xi_2^*, \dots, \xi_m^*, \dots \rangle$.

The following theorem shows that ℓ_p spaces can be ordered with respect to the set relation " \subset ". That is, if a sequence belongs to ℓ_1 , then it belongs to ℓ_2 , etc. For example, $\langle \frac{1}{n} \rangle \notin \ell_1$, but $\langle \frac{1}{n} \rangle \in \ell_p$ for $p > 1$.

Theorem 239 *If $1 < p < q < \infty$, then $\ell_1 \subset \ell_p \subset \ell_q \subset \ell_\infty$ and $\|\langle x_n \rangle\|_q \leq \|\langle x_n \rangle\|_p$.*

Proof. Start with $\ell_p \subset \ell_\infty$. Let $x \in \ell_p$ (i.e. $\sum_{i=1}^{\infty} |x_i|^p < \infty$) so that $\langle |x_n| \rangle$ is bounded. Then $\sup \{|x_n|, n \in \mathbb{N}\} < \infty$ so that $x \in \ell_\infty$. We also have

$$\forall j, |x_j|^p \leq \sum_{i=1}^{\infty} |x_i|^p \iff |x_j| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

Therefore $\sup \{|x_j|, j \in \mathbb{N}\} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$ or $\|x\|_\infty \leq \|x\|_p$.

Next we show $\ell_p \subset \ell_q$ for $p < q$ and $1 \leq p, q < \infty$.

$$\begin{aligned} (\|x\|_q)^q &= \sum_{i=1}^{\infty} |x_i|^q = (\|x\|_p)^q \sum_{i=1}^{\infty} \left(\frac{|x_i|}{\|x\|_p} \right)^q \\ &\leq (\|x\|_p)^q \sum_{i=1}^{\infty} \left(\frac{|x_i|}{\|x\|_p} \right)^p \\ &= (\|x\|_p)^q \frac{\|x\|_p^p}{\|x\|_p^p} \\ &= (\|x\|_p)^q. \end{aligned}$$

where the inequality follows since $\frac{|x_i|}{\|x\|_q} \leq 1$ and $q > p$. Taking the q -root of the above inequality gives $\|x\|_q \leq \|x\|_p$. Now if $x \in \ell_p$ (i.e. $\|x\|_p < \infty$), then $\|x\|_q < \infty$ and $x \in \ell_q$. ■

Example 240 Note that the inclusion $\ell_p \subset \ell_q$ for $p < q$ is strict. To see this, consider the sequence $\langle x_n \rangle = \left\langle \frac{1}{n^{\frac{1}{p}}} \right\rangle_{n=1}^{\infty}$. It is simpler to work with the

p th power of a norm to avoid using the p th root. Hence, take $\left(\left\| \frac{1}{n^{\frac{1}{p}}} \right\|_p \right)^p = \sum_{n=1}^{\infty} \left(\frac{1}{n^{\frac{1}{p}}} \right)^p = \sum_{n=1}^{\infty} \frac{1}{n}$ which is infinitely large (we showed this in the example of a harmonic series). Hence, $\left\langle \frac{1}{n^{\frac{1}{p}}} \right\rangle_{n=1}^{\infty} \notin \ell_p$. However $\left\langle \frac{1}{n^{\frac{1}{p}}} \right\rangle_{n=1}^{\infty} \in \ell_q$. To see this, $\left(\left\| \frac{1}{n^{\frac{1}{p}}} \right\|_q \right)^q = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{q}{p}}}$ where $\frac{q}{p} > 1$, this series is bounded (this can be shown by using the integral criterion - See Bartle).

The fundamental difference between ℓ_p with $1 \leq p < \infty$ and ℓ_∞ is the behavior of their tails. While it's easy to see that for $1 \leq p < \infty$ if $x \in \ell_p$

then $\lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} |x_i|^p = 0$. It is not true in ℓ_{∞} . For instance the sequence $\langle x_i \rangle = \langle 1, 1, \dots, 1, \dots \rangle \in \ell_{\infty}$ but the norm of its tail is 1. This is the reason why there are properties of ℓ_{∞} that are different from those of ℓ_p , $1 \leq p < \infty$. One of these properties is separability (i.e. the existence of a dense countable subset.)

Theorem 241 ℓ_p is separable for $1 \leq p < \infty$.

Proof. Let $\{e_i, i \in N\}$ be a basis of unit vectors. Then the set of all linear combinations $H = \{\sum_{i=1}^n \alpha_i e_i, \alpha_i \in \mathbb{Q}\}$ is countable and dense in ℓ_p because if $x = (x_1, x_2, \dots) \in \ell_p$, then the tail of x (given by)

$$\left\| x - \sum_{i=1}^n x_i e_i \right\|_p = \left(\sum_{i=n+1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \rightarrow 0.$$

Thus, x is approximated by an element of H . ■

Theorem 242 ℓ_{∞} is not separable.

Proof. Let S be the set of all sequences containing only 0 and 1; that is $S = \{0, 1\}^N$. Clearly $S \subset \ell_{\infty}$ and if $x = \langle x_n \rangle$, $y = \langle y_n \rangle$ are two distinct elements of S , then $\|x - y\|_{\infty} = 1$. Hence $B_{\frac{1}{2}}(x) \cap B_{\frac{1}{2}}(y) = \emptyset$ for any $x, y \in \ell_{\infty}, x \neq y$. Let A be a dense set in ℓ_{∞} . Then for $\varepsilon = \frac{1}{2}$ and given $x \in S \subset \ell_{\infty}$, there exists an element $a \in A$ such that $\|x - a\|_{\infty} < \frac{1}{2}$. Because S is uncountable A must be uncountable, thus any dense set in ℓ_{∞} must be uncountable. ■

4.6 Continuous Functions

Now we return to another important topological concept in mathematics that is employed extensively in economics. Before defining continuity, we amend Definition 49 of a function in Section 5.2 in terms of general metric spaces.

Definition 243 A function f from a metric space (X, d_X) into a metric space (Y, d_Y) is a rule that associates to each $x \in X$ a unique $y \in Y$.

Definition 244 Given metric spaces (X, d_X) and (Y, d_Y) , the function $f : X \rightarrow Y$ is (pointwise) **continuous at x** if, $\forall \varepsilon > 0, \exists \delta(\varepsilon, x) > 0$ such that if $d_X(x', x) < \delta(\varepsilon, x)$, then $d_Y(f(x), f(x')) < \varepsilon$. The function is **continuous** if it is continuous at each $x \in X$. See Figure 4.6.1.

Example 245 Let $(X, d_X) = ((-\infty, 0) \cup (0, \infty), |\cdot|)$, $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, and define

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Then $f : X \rightarrow Y$ is continuous on (X, d_X) . See Figure 4.6.2.

Example 246 Let $(X, d_X) = (\mathbb{R}, |\cdot|)$, $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, and define $f(x) = bx$, $b \in \mathbb{R} \setminus \{0\}$. Then $f : X \rightarrow Y$ is continuous on (X, d_X) since we can simply let $\delta(\varepsilon, x) = \frac{\varepsilon}{|b|}$. Then, for any $\varepsilon > 0$, if $|x' - x| < \delta(\varepsilon, x)$ we have $|bx' - bx| = |b||x' - x| < \varepsilon$. Notice that in the case of linear functions, δ is independent of x . Figure 4.6.3.

Example 247 Let $(X, d_X) = (\mathbb{R} \setminus \{0\}, |\cdot|)$, $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, and define $f(x) = \frac{1}{x}$. For any $x \in X$, then

$$|f(x') - f(x)| = \left| \frac{1}{x'} - \frac{1}{x} \right| = \frac{|x' - x|}{|xx'|}.$$

We wish to find a bound for the coefficient of $|x' - x|$ which is valid around 0. If $|x' - x| < \frac{1}{2}|x|$, then $\frac{1}{2}|x| < |x'|$ in which case

$$|f(x') - f(x)| \leq \frac{2}{|x|^2}|x' - x|.$$

In this case, $\delta(\varepsilon, x) = \inf\{\frac{1}{2}|x|, \frac{1}{2}\varepsilon|x|^2\}$. Figure 4.6.4.

There is an equivalent way to define pointwise continuity in terms of the inverse image (Definition 53) and in terms of sequences.

Theorem 248 Given metric spaces (X, d_X) and (Y, d_Y) , the following statements are equivalent: (i) function $f : X \rightarrow Y$ is continuous; (ii) if for each open subset V of Y , the set $f^{-1}(V)$ is an open subset of X ; and (iii) if for every convergent sequence $x_i \rightarrow x$ in X , the sequence $f(x_i) \rightarrow f(x)$.

Proof. (Sketch)(ii) \Rightarrow (i) Any ε -ball around $f(x)$ is open so there is a δ -ball around x inside $f^{-1}(B_\varepsilon(f(x)))$. (iii) \Rightarrow (ii) If not, then there is an $x \in f^{-1}(V)$ such that for any $\frac{1}{n}$ neighborhood of it, we can find a point x_n such that $f(x_n) \notin V$. But $\langle x_n \rangle$ contradicts (iii). (i) \Rightarrow (iii) From (i) for x_n close enough to x , $f(x_n)$ will be as close to $f(x)$ as we want, so that $f(x_n) \rightarrow f(x)$. ■

The previous two examples go against “conventional wisdom” that the graph of a continuous function is not interrupted and may raise the question of the existence of a function that is not continuous. The following example provides such a function.

Example 249 Let $(X, d_X) = (\mathbb{R}, |\cdot|)$, $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, and define¹⁷

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} .$$

Then $f^{-1}((-\frac{1}{2}, \frac{1}{2})) = \{0\}$, the inverse image of an open set is closed, therefore this function is not continuous in (X, d) . See Figure 4.6.5.

Next we show that the composition of continuous functions preserves continuity.

Theorem 250 Given metric spaces (X, d_X) , (Y, d_Y) , and (Z, d_Z) , and continuous functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then $h : X \rightarrow Z$ given by $h = g \circ f$ is continuous.

Proof. Let $U \subset Z$ be open. Then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X . But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}$. ■

It follows that certain simple operations with continuous functions preserve continuity.

Theorem 251 Given a metric space (X, d_X) and a normed vector space (Y, d_Y) , and continuous functions $f : X \rightarrow Y$ and $g : X \rightarrow Y$, then the following are also continuous: (i) $f \pm g$; (ii) $f \cdot g$; (iii) $\frac{f}{g}$; (iv) $|f|$.

Exercise 4.6.1 Prove Theorem 251.

¹⁷This is known as the “sgn” function.

It should be emphasized that Theorem 250 does not say that if f is continuous and U is open in X then the image $f(U) = \{f(x), x \in U\}$ is open in Y .

Example 252 Let $(X, d_X) = (\mathbb{R}, |\cdot|)$, $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, and define $f(x) = x^2$. Then $f((-1, 1)) = [0, 1)$ is the image of an open set which is not open. See Figure 4.6.7.

Therefore continuity does not preserve openness. It does not preserve closedness either as the next example shows.

Example 253 Let $(X, d_X) = (\mathbb{R} \setminus \{0\}, |\cdot|)$, $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, and define $f(x) = \frac{1}{x}$. Then $f([1, \infty)) = (0, 1]$ is the image of a closed set which is not closed. See Figure 4.6.8.

There are, however, important properties of a set which are preserved under continuous mapping. The next subsections establish this.

4.6.1 Intermediate value theorem

Theorem 254 (Preservation of Connectedness) *The image of a connected space under a continuous function is connected.*

Proof. Let $f : X \rightarrow Y$ be a continuous function on X and let X be connected. We wish to prove that $Z = f(X)$ is connected. Assume the contrary. Then there exists open disjoint sets A and B such that $Z = (A \cap Z) \cup (B \cap Z)$ and $(A \cap Z), (B \cap Z)$ is a separation of Z into two disjoint, non-empty sets in Z . Then $f^{-1}(A \cap Z) = f^{-1}(A) \cap f^{-1}(Z) = f^{-1}(A) \cap X = f^{-1}(A)$ and $f^{-1}(B \cap Z) = f^{-1}(B)$ are disjoint sets whose union is $X (= f^{-1}(A \cap Z) \cup f^{-1}(B \cap Z))$. They are open in X because f is continuous and non-empty because $f : X \rightarrow f(X)$ is a surjection. Therefore $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of X which contradicts the assumption that X is connected. ■

In the special case where the metric space $(Y, d_Y) = (\mathbb{R}, |\cdot|)$ then the corollary of this theorem is the well-known Intermediate Value Theorem.

Corollary 255 (Intermediate Value Theorem) *Let $f : X \rightarrow \mathbb{R}$ be a continuous function of a connected space X into \mathbb{R} . If $a, b \in X$ and if $r \in Y$ such that $f(a) \leq r \leq f(b)$, then $\exists c \in X$ such that $f(c) = r$. See Figure 4.6.9.*

Exercise 4.6.2 Prove Corollary 255.

Note that it is connectedness that is required for the Intermediate value theorem and not compactness.

Example 256 Let $(X, d_X) = ([-2, -1] \cup [1, 2], |\cdot|)$, $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, and define

$$f(x) = \begin{cases} 1 & \text{if } x \in [1, 2] \\ -1 & \text{if } x \in [-2, -1] \end{cases}.$$

Then $f : X \rightarrow Y$ is continuous on the compact set X but for $r = 0$, there doesn't exist $c \in X$ such that $f(c) = 0$.

A nice one dimensional example of how important the intermediate value theorem is for economics, is the following fixed point theorem.

Corollary 257 (One Dimensional Brouwer) Let $f : [a, b] \rightarrow [a, b]$ be a continuous function. Then f has a fixed point.

Proof. Let $g : [a, b] \rightarrow \mathbb{R}$ be defined by $g(x) = f(x) - x$. Clearly $g(a) = f(a) - a \geq 0$ since $f(a) \in [a, b]$ and $g(b) = f(b) - b \leq 0$ for the same reason. Since $g(x)$ is a continuous function¹⁸ with $g(b) \leq 0 \leq g(a)$, we know by the Intermediate Value Theorem 255 that $\exists \bar{x} \in [a, b]$ such that $g(\bar{x}) = 0$ or equivalently that $f(\bar{x}) = \bar{x}$. ■

The proof is illustrated in Figure 4.6.10. For a more general version of this proof, see Section 4.8.

The next series of examples shows how connectedness of \mathbb{R}_+ can be used to construct a continuous “utility” function $u(x)$ that represents a preference relation \succsim . Before establishing this, however, we need to define continuity in terms of relations.

Definition 258 The preference relation \succsim on X is **continuous** if for any sequence of pairs $\langle (x_n, y_n) \rangle_{n=1}^{\infty}$ with $x_n \succsim y_n \forall n$, $x = \lim_{n \rightarrow \infty} x_n$, and $y = \lim_{n \rightarrow \infty} y_n$, then $x \succsim y$.

¹⁸To see $g(x)$ is a continuous function, we must show $\forall \varepsilon > 0$ and $x, y \in [a, b]$, $\exists \delta_g > 0$ such that if $|x - y| < \delta_g$ then $|g(x) - g(y)| < \varepsilon$. But

$$|g(x) - g(y)| = |(f(x) - f(y)) - (x - y)| \leq |f(x) - f(y)| + |x - y|$$

by the triangle inequality. Continuity of f implies $\forall \varepsilon > 0, \exists \delta_f > 0$ such that $|x - y| < \delta_f$ and $|f(x) - f(y)| < \varepsilon$. Thus, let if we let $\delta_g = \min\{\delta_f, \varepsilon\}/2$, then $|g(x) - g(y)| < \varepsilon$.

An equivalent way to state this notion of continuity is that $\forall x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ and the lower contour set $\{y \in X : x \succsim y\}$ are both closed; that is, for any $\langle y_n \rangle_{n=1}^{\infty}$ such that $x \succsim y_n, \forall n$ and $y = \lim y_n$, we have $x \succsim y$ (just let $x_n = x, \forall n$).

There are some preference relations that are not continuous as the following example shows.

Example 259 *Lexicographic preferences* (on $X = \mathbb{R}_+^2$) are defined in the following way: $x \succsim y$ if either “ $x_1 > y_1$ ” or “ $x_1 = y_1$ and $x_2 \geq y_2$ ”. To see they are not continuous, consider the sequence of bundles $\langle x_n = (\frac{1}{n}, 0) \rangle$ and $\langle y_n = (0, 1) \rangle$. For every n we have $x_n \succ y_n$. But $\lim_{n \rightarrow \infty} y_n = (0, 1) \succ (0, 0) = \lim_{n \rightarrow \infty} x_n$. That is, as long as the first component of x is larger than that of y , x is preferred to y even if y_2 is much larger than x_2 . But as soon as the first components become equal, only the second components are relevant so that the preference ranking is reversed at the limit points.

Now we establish that we can “construct” a continuous utility function.

Example 260 *If the rational preference relation \succsim on X is continuous, then there is a continuous utility function $u(x)$ that represents \succsim . To see this, by continuity of \succsim , we know that the upper and lower contour sets are closed. Then the sets $A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$ and $A^- = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$, where e is the unit vector, are nonempty and closed. By completeness of \succsim , $\mathbb{R}_+ \subset (A^+ \cup A^-)$. The nonemptiness and closedness of A^+ and A^- , along with the fact that \mathbb{R}_+ is connected, imply $A^+ \cup A^- \neq \emptyset$. Thus, $\exists \alpha$ such that $\alpha e \sim x$. By monotonicity of \succsim , $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Hence, there can be at most one scalar satisfying $\alpha e \sim x$. This scalar is $\alpha(x)$, which we take as the utility function.*

4.6.2 Extreme value theorem

The next result is one of the most important ones for economists we will come across in the book.

Theorem 261 (Preservation of Compactness) *The image of a compact set under a continuous function is compact.*

Proof. Let $f : X \rightarrow Y$ be a continuous function on X and let X be compact. Let \mathcal{G} be an open covering of $f(X)$ by sets open in Y . The

collection $\{f^{-1}(G), G \in \mathcal{G}\}$ is a collection of sets covering X . These sets are open in X because f is continuous. Hence finitely many of them, say $f^{-1}(G_1), \dots, f^{-1}(G_n)$ cover X . Then the sets G_1, \dots, G_n cover $f(X)$. ■

Again in the special case where $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, a direct consequence of this theorem is the well known Extreme Value Theorem of calculus.¹⁹

Corollary 262 (Extreme Value Theorem) *Let $f : X \rightarrow \mathbb{R}$ be a continuous function of a compact space X into \mathbb{R} . Then $\exists c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$. $f(c)$ is called the minimum and $f(d)$ is called the maximum of f on X .*

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact. We show that A has a largest element M and a smallest element m . Then since m and M belong to $f(X)$, we must have $m = f(c)$ and $M = f(d)$ for some points c and d of X .

If A has no largest element, then the collection $\{(-\infty, a), a \in A\}$ forms an open covering of A . Since A is compact, some finite subcollection $\{(-\infty, a_1), \dots, (-\infty, a_n)\}$ covers A . Let $a_M = \max\{a_1, \dots, a_n\}$ then $a_M \in A$ belongs to none of these sets, which contradicts the fact that they cover A .

A similar argument can be used to show that A has a smallest element. ■

Exercise 4.6.3 *Let $X = [0, 1)$ and $f(x) = x$. Why doesn't a maximum exist? See Figure 4.6.11.*

4.6.3 Uniform continuity

One might believe from part (iii) of the Theorem 248 that if $\langle x_n \rangle$ is Cauchy and if f is continuous, then $\langle f(x_n) \rangle$ is also Cauchy. The following examples show this is false if f is pointwise continuous.

Example 263 *Take the sequence $\langle x_n \rangle = \left\langle \frac{(-1)^n}{n} \right\rangle$ and consider the function f defined in Example 245. While $\langle x_n \rangle$ is Cauchy in $(-\infty, 0) \cup (0, \infty)$, $\langle f(x_n) \rangle = \langle -1, 1, -1, 1, \dots \rangle$ which is not Cauchy. See Figure 4.6.12.*

Example 264 *Let $f(x) = \frac{1}{x}$ on $(0, 1]$ which was shown to be pointwise continuous in Example 247. Consider the Cauchy sequence $\langle \frac{1}{n} \rangle$ on $(0, 1]$. It is clear that $\langle f(x_n) \rangle = \langle n \rangle$, which is obviously not Cauchy. See Figure 4.6.13.*

¹⁹Sometimes this is called the Maximum and Minimum Value Theorem. Since in the next section we will introduce the Maximum Theorem, we choose the above terminology.

For the above intuition to hold, we need a stronger concept of continuity.

Definition 265 Given metric spaces (X, d_X) and (Y, d_Y) , the function $f : X \rightarrow Y$ is **uniformly continuous** if $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ such that $\forall x, x' \in X$ with $d_X(x', x) < \delta(\varepsilon)$, then $d_Y(f(x), f(x')) < \varepsilon$.

While this definition looks similar to that of pointwise continuity in Definition 244, the difference is that while δ generally depends on both ε and x in the case of pointwise continuity, it is independent of x in case of uniform continuity.

Theorem 266 Given metric spaces (X, d_X) and (Y, d_Y) , let the function $f : X \rightarrow Y$ be uniformly continuous. If $\langle x_n \rangle$ is a Cauchy sequence in X , then $\langle f(x_n) \rangle$ is a Cauchy sequence in Y .

Proof. Let $\langle x_n \rangle$ be a Cauchy sequence in X . Because $f : X \rightarrow Y$ is uniformly continuous then $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ such that $\forall x, x' \in X$ with $d_X(x', x) < \delta(\varepsilon)$, then $d_Y(f(x), f(x')) < \varepsilon$. Since $\langle x_n \rangle$ is Cauchy for given $\delta(\varepsilon) > 0$ there $\exists N$ such that $\forall m, n \in \mathbb{N}$ with $m, n > N$ then $d_X(x_m, x_n) < \delta(\varepsilon)$. But then $d_Y(f(x_m), f(x_n)) < \varepsilon$. Hence $\langle f(x_n) \rangle$ is a Cauchy sequence in Y .

■

According to this theorem the functions in Examples 263 and 264 are not uniformly continuous. Notice that the domains of each of the functions in the examples are not compact in $(\mathbb{R}, |\cdot|)$. Let's consider another example.

Example 267 Let $f : [0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = x^2$. This function is continuous on \mathbb{R} . Is it uniformly continuous? No. We show this by finding an $\varepsilon > 0$ such that $\forall \delta > 0, \exists x_1, x_2$ such that $d_X(x_1, x_2) < \delta$ and $d_Y(f(x_1), f(x_2)) \geq \varepsilon$. Let $\varepsilon = 2$ and take any $\delta > 0$. Then $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Define $x_1 = n + \frac{1}{n}$ and $x_2 = n$. Then $d_X(x_1, x_2) = (n + \frac{1}{n} - n) = \frac{1}{n} < \delta$ and $d_Y(f(x_1), f(x_2)) = (n + \frac{1}{n})^2 - n^2 = 2 + \frac{1}{n^2} > 2$. Notice that the domain of this function $[0, \infty)$ is not compact in $(\mathbb{R}, |\cdot|)$.

If the domain of a continuous function is compact, then the function is also uniformly continuous as the following theorem asserts.

Theorem 268 (Uniform Continuity Theorem) Let $f : X \rightarrow Y$ be a continuous function of a compact metric space (X, d_X) to the metric space (Y, d_Y) . Then f is uniformly continuous.

Proof. (Sketch) For a given $\varepsilon > 0$, by continuity of f around any $x \in X$ we can find a $\delta(\frac{1}{2}\varepsilon, x)$ -ball such that for $x' \in B_{\delta(\frac{1}{2}\varepsilon, x)}(x)$ we have $d_Y(f(x), f(x')) < \frac{1}{2}\varepsilon$. Since the collection of such open balls is an open covering of X and X is compact, there exists a finite (say n) subcover of them. Then for $x, x' \in X$ such that $d_X(x', x) < \delta(\varepsilon) = \frac{1}{2} \min\{\delta(\frac{1}{2}\varepsilon, x_1), \dots, \delta(\frac{1}{2}\varepsilon, x_n)\}$, there exists k such that $x \in B_{\delta(\frac{1}{2}\varepsilon, x_k)}(x_k)$ and $x' \in B_{\delta(\frac{1}{2}\varepsilon, x_k)}(x_k)$. Therefore by the triangle inequality $d_Y(f(x), f(x')) < \varepsilon$. ■

The number $\delta(\varepsilon)$ that we constructed in the proof of Theorem 268, is called the **Lebesgue number** of the covering \mathcal{G} .

Exercise 4.6.4 Why is $f(x) = \frac{1}{x}$ not uniformly continuous on $X = (0, 1]$ but it is on $[10^{-1000}, 1]$?

4.7 Hemicontinuous Correspondences

Many problems in economics result in set-valued mappings or correspondences as defined in Section 2.3. For instance, if preferences are linear, a household's demand for goods may be described by a correspondence and in game theory we consider best response correspondences.

Before defining hemicontinuity, we amend Definition 48 of a correspondence in Section 2.3 in terms of general metric spaces.

Definition 269 A **correspondence** Γ from a metric space (X, d_X) into a metric space (Y, d_Y) is a rule that associates to each $x \in X$ a subset $\Gamma(x) \in Y$. Its **graph** is the set $A = \{(x, y) \in X \times Y : y \in \Gamma(x)\}$ which we will denote $Gr(\Gamma)$. The **image** of a set $D \subset X$, denoted $\Gamma(D) \subset Y$, is the set $\Gamma(D) = \cup_{x \in D} \Gamma(x)$. A correspondence is **closed valued** at x if the image set $\Gamma(x)$ is closed in Y . A correspondence is **compact valued** at x if the image set $\Gamma(x)$ is compact in Y . See Figure 4.7.1.

Unlike a (single-valued) function, there are two ways to define the inverse image of a correspondence Γ of subset D .

Definition 270 For $\Gamma : X \rightrightarrows Y$ and any subset $D \subset Y$ we define the **inverse image** (also lower or weak) as $\Gamma^{-1}(D) = \{x \in X : \Gamma(x) \cap D \neq \emptyset\}$ and the **core** (also upper or strong inverse image) $\Gamma^{+1}(D) = \{x \in X : \Gamma(x) \subset D\}$.

It is clear that $\Gamma^{+1}(D) \subset \Gamma^{-1}(D)$. Also observe that

$$\begin{aligned}\Gamma^{+1}(Y \setminus D) &= X - \Gamma^{-1}(D) \text{ and} \\ \Gamma^{-1}(Y \setminus D) &= X \setminus \Gamma^{+1}(D).\end{aligned}$$

See Figure 4.7.2. These two types of inverse image naturally coincide when Γ is single-valued.

To make the notion of correspondence clearer we present a number of examples (see Figure 4.7.3a-3f).

Example 271 $\Gamma : [0, 1] \rightarrow [0, 1]$ defined by $\Gamma(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2} \\ \{0, 1\} & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x > \frac{1}{2} \end{cases} .$

Example 272 $\Gamma : [0, 1] \rightarrow [0, 1]$ defined by $\Gamma(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2} \\ [0, 1] & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x > \frac{1}{2} \end{cases} .$

Example 273 $\Gamma : [0, 1] \rightarrow [0, 1]$ defined by $\Gamma(x) = [x, 1]$.

Example 274 $\Gamma : [0, 1] \rightarrow [0, 1]$ defined by $\Gamma(x) = \begin{cases} [0, \frac{1}{2}] & \text{if } x \neq \frac{1}{2} \\ [0, 1] & \text{if } x = \frac{1}{2} \end{cases} .$

Example 275 $\Gamma : [0, 1] \rightarrow [0, 1]$ defined by $\Gamma(x) = \begin{cases} [0, 1] & \text{if } x \neq \frac{1}{2} \\ [0, \frac{1}{2}] & \text{if } x = \frac{1}{2} \end{cases} .$

Example 276 $\Gamma : [0, \infty) \rightarrow \mathbb{R}$ defined by $\Gamma(x) = [e^{-x}, 1]$.

We next define a set valued version of continuity.

Definition 277 Given metric spaces (X, d_X) and (Y, d_Y) , the correspondence $\Gamma : X \rightarrow Y$ is **lower hemicontinuous (lhc)** at $x \in X$ if $\Gamma(x)$ is non-empty and if for every open set $V \subset Y$ with $\Gamma(x) \cap V \neq \emptyset$, there exists a neighborhood U of x such that $\Gamma(x') \cap V \neq \emptyset$ for every $x' \in U$. The correspondence is **lower hemicontinuous** if it is lhc at each $x \in X$.²⁰ See Figure 4.7.4.

²⁰There are various names given to this concept. In many math books, this is called *semicontinuity*.

Note that the correspondences presented in Examples 273, 275, and 276 are lhc.

As in the case of continuity of a function, there are equivalent characterizations of lhc in terms of open (closed) sets or sequences as the next theorem shows.

Theorem 278 *Given metric spaces (X, d_X) and (Y, d_Y) , for a correspondence $\Gamma : X \rightarrow Y$ the following statements are equivalent. (i) Γ is lhc; (ii) $\Gamma^{-1}(V)$ is open in X whenever $V \subset Y$ is open in Y ; (iii) $\Gamma^{+1}(U)$ is closed in X whenever $U \subset Y$ is closed in Y ; and (iv) $\forall x \in X, \forall y \in \Gamma(x)$ and every sequence $\langle x_n \rangle \rightarrow x, \exists N$ such that $\langle y_n \rangle \rightarrow y$ and $y_n \in \Gamma(x_n), \forall n \geq N$.*

Proof. (i) \iff (ii) Let V be open in Y , $\Gamma^{-1}(V) = \{x \in X : \Gamma(x) \cap V \neq \emptyset\}$ and take $x_0 \in \Gamma^{-1}(V)$. Since Γ is lhc at x_0 then $\exists U$ open such that, $x_0 \in U, \Gamma(x') \cap V \neq \emptyset$ for every $x' \in U$. Hence $U \subset \Gamma^{-1}(V)$ so that $\Gamma^{-1}(V)$ is open.

(ii) \iff (iii) follows immediately from $X \setminus \Gamma^{-1}(U) = \Gamma^{+1}(Y \setminus U)$.

(i) \iff (iv) First start with (\implies) . Let $\langle x_n \rangle \rightarrow x$ and fix an arbitrary point $y \in \Gamma(x)$. For each $k \in \mathbb{N}$, $B_{\frac{1}{k}}(y) \cap \Gamma(x) \neq \emptyset$. Since Γ is lhc at $x, \forall k$ there exists an open set U_k of x such that $\forall x'_k \in U_k$ we have $\Gamma(x'_k) \cap B_{\frac{1}{k}}(y) \neq \emptyset$. Since $\langle x_n \rangle \rightarrow x, \forall k$ we can find n_k such that $x_n \in U_k, \forall n \geq n_k$ and they can be assigned so that $n_{k+1} > n_k$. Also, since $x_n \in U_k, \forall n \geq n_k$, then $\Gamma(x_n) \cap B_{\frac{1}{k}}(y) \neq \emptyset$. Hence we can construct a companion sequence $\langle y_n \rangle$, with y_n chosen from the set $\Gamma(x_n) \cap B_{\frac{1}{k}}(y)$ for each $n \geq n_k$. As k , and hence n , increases the radius of the balls $B_{\frac{1}{k}}(y)$ shrinks to zero, implying $\langle y_n \rangle \rightarrow y$. Next we prove (\impliedby) . In this case, it is sufficient to prove the contrapositive. Assume Γ is not lhc at x . Then $\exists V$ with $\Gamma(x) \cap V \neq \emptyset$ such that every neighborhood U of x contains a point x'_u with $\Gamma(x'_u) \cap V = \emptyset$. Taking a sequence of such neighborhoods, $U_n = B_{\frac{1}{n}}(x)$ and a point in each of them, we obtain a sequence $\langle x_n \rangle \rightarrow x$ by construction and has the property $\Gamma(x_n) \cap V = \emptyset$. Hence every companion sequence $\langle y_n \rangle$ with $y_n \in \Gamma(x_n)$ is contained in the complement of V , and if $\langle y_n \rangle \rightarrow y$ then y is contained in the complement of V since $Y \setminus V$ is closed. Thus no companion sequence of $\langle x_n \rangle$ can converge to a point in V . ■

Thus, Γ is lhc at x if any $y \in \Gamma(x)$ can be approached by a sequence from both sides. Also, if the correspondence F is a function, then $F^{-1}(U)$ is the inverse image of a function so (ii) states F is lhc iff F is continuous.

Definition 279 Given metric spaces (X, d_X) and (Y, d_Y) , the correspondence $\Gamma : X \rightrightarrows Y$ is **upper hemicontinuous (uhc)** at $x \in X$ if $\Gamma(x)$ is non-empty and if for every open set $V \subset Y$ with $\Gamma(x) \subset V$, there exists a neighborhood U of x such that $\Gamma(x') \subset V$ for every $x' \in U$. The correspondence is **upper hemicontinuous** if it is uhc at each $x \in X$. See Figure 4.7.5.

The correspondences presented in Examples 271-276 are uhc.

Again, uhc can be characterized in terms of open (closed) sets or sequences.

Theorem 280 Given metric spaces (X, d_X) and (Y, d_Y) , for a correspondence $\Gamma : X \rightrightarrows Y$ the following statements are equivalent: (i) Γ is uhc; (ii) $\Gamma^{+1}(V)$ is open in X whenever $V \subset Y$ is open in Y ; (iii) $\Gamma^{-1}(U)$ is closed in X whenever $U \subset Y$ is closed in Y ; and if Γ is compact valued, then (iv) for every sequence $\langle x_n \rangle \rightarrow x$ and every sequence $\langle y_n \rangle$ such that $y_n \in \Gamma(x_n)$, $\forall n$, there exists a convergent subsequence $\langle y_{g(n)} \rangle \rightarrow y$ and $y \in \Gamma(x)$.

Proof. (Sketch)(i) and Γ compact \Rightarrow (iv). First, we must show that the companion sequence $\langle y_n \rangle$ is bounded. Since $\langle y_n \rangle$ is bounded, it is contained in a compact set so that by Theorem 193, there exists a convergent subsequence. Finally, we must show that the limit of this subsequence is in $\Gamma(x)$.

(i) \Leftarrow (iv) Again, it is sufficient to prove the contrapositive; If Γ is not uhc at x , then there is no subsequence converging to a point in $\Gamma(x)$. ■

Exercise 4.7.1 Finish the proof of Theorem 280. ²¹

???Thus, Γ is uhc at x if any $y \in \Gamma(x)$ can be approached by a sequence from ????

If the correspondence F is a function, then $F^{+1}(U) = F^{-1}(U)$ is the inverse image of the function and so by (ii), F is uhc iff F is continuous.

Each type of hemicontinuity can be interpreted in terms of the restrictions of the “size” of the set $\Gamma(x)$ as x changes.

- Suppose Γ is uhc at x and fix $V \supset \Gamma(x)$. As we move from x to a nearby point x' , the set V gives an “upper bound” on the size of $\Gamma(x')$

²¹de la Fuente (Theorem 11.2, p. 110).

since we require $\Gamma(x') \subset V$. Hence uhc requires the image set $\Gamma(x)$ does not “explode” with small changes in x , but allows it to suddenly “implode”.

- Suppose Γ is lhc at x . As we move from x to a nearby point x' , the set V gives a “lower bound” on the size of $\Gamma(x')$ since we require $\Gamma(x') \cap V \neq \emptyset$. Hence lhc requires the image set $\Gamma(x)$ does not “implode” with small changes in x , but allows it to suddenly “explode”.

Definition 281 *Given metric spaces (X, d_X) and (Y, d_Y) , a correspondence $\Gamma : X \rightarrow Y$ is **continuous** at $x \in X$ if it is both lhc and uhc at x .*

The correspondences in Examples 273 and 276 are continuous.

Example 282 *Consider the following example of a best response correspondence derived from game theory. The game is played between two individuals who can choose between two actions, say go up (U) or go down (D). If both choose U or both choose D , they meet. If one chooses U and the other chooses D , they don't meet. Meetings are pleasurable and yield each player payoff 1, while if they don't meet they receive payoff 0. This is known as a coordination game. The players choose probability distributions over the two actions: say player 1 chooses U with probability p and D with probability $1-p$ while player 2 chooses U with probability q and D with probability $1-q$. We represent this game in “normal form” by the matrix in Table 4.7.1. Agent 1's payoff from playing U with probability p while his opponent is playing U with probability q is denoted $\pi_1(p, q)$ and given by*

$$\begin{aligned}\pi_1(p, q) &= p \cdot [q \cdot 1 + (1 - q) \cdot 0] + (1 - p) \cdot [q \cdot 0 + (1 - q) \cdot 1] \\ &= 1 - q - p + 2pq\end{aligned}$$

Agent 1 chooses $p \in [0, 1]$ to maximize $\pi_1(p, q)$. We call this choice a best response correspondence $p^*(q)$. It is simple to see that: for any $q < \frac{1}{2}$, profits are decreasing in p so that $p^* = 0$ is a best response, for any $q > \frac{1}{2}$, profits are increasing in p so that $p^* = 1$ is a best response, and at $q = \frac{1}{2}$, profits are independent of p so that any choice of $p^* \in [0, 1]$ is a best response.²²

²²To see this, note that $\frac{d\pi}{dp} = 2q - 1$ so that

$$\begin{aligned}\frac{d\pi}{dp} &< 0 & \text{if } q < \frac{1}{2} \\ \frac{d\pi}{dp} &= 0 & \text{if } q = \frac{1}{2} \\ \frac{d\pi}{dp} &> 0 & \text{if } q > \frac{1}{2}\end{aligned} .$$

Obviously, $p^*(q)$ is a correspondence. It is not lhc at $q = \frac{1}{2}$ since if we let $V = (\frac{1}{4}, \frac{3}{4})$, then $p^*(\frac{1}{2}) \cap V \neq \emptyset$ and there exists no neighborhood U around $\frac{1}{2}$ such that $p^*(q) \cap V \neq \emptyset$ for $q \in U$ (e.g. for any $\frac{1}{2} \geq \varepsilon > 0$, $p^*(\frac{1}{2} - \varepsilon) = 0 \notin (\frac{1}{4}, \frac{3}{4})$ and $p^*(\frac{1}{2} + \varepsilon) = 1 \notin (\frac{1}{4}, \frac{3}{4})$). It is, however, uhc at $q = \frac{1}{2}$ since we must take $V = (a, b)$ with $a < 0$ and $b > 1$ to satisfy $p^*(\frac{1}{2}) = [0, 1] \subset V$. But then there exist many neighborhoods U around $\frac{1}{2}$ such that $p^*(q) \subset V$ for $q \in U$ (e.g. for any $\frac{1}{2} \geq \varepsilon > 0$, $p^*(\frac{1}{2} - \varepsilon) = 0 \in (a, b)$ and $p^*(\frac{1}{2} + \varepsilon) = 1 \in (a, b)$). See Figure 4.7.6. Finally, you should recognize that this game is symmetric so that agent 2's payoffs and hence best response correspondence is identical to that of agent 1.

Table 4.7.1

player	2	q	$1 - q$
1		U	D
p	U	1, 1	0, 0
$1 - p$	D	0, 0	1, 1

Just as it was cumbersome to apply Definition 185 to establish compactness, it is similarly cumbersome to apply Definitions 277 and 279 to establish hemicontinuity. In the case of compactness, we provided simple sufficient conditions (e.g. the Heine-Borel Corollary 194). Here we supply another set of simple sufficient conditions to establish hemicontinuity.

Theorem 283 *Let $\Gamma : X \rightarrow Y$ be a non-empty valued correspondence and let A be its graph. If (i) A is convex and (ii) for any bounded set $\hat{X} \subset X$, there is a bounded set $\hat{Y} \subset Y$ such that $\Gamma(x) \cap \hat{Y} \neq \emptyset, \forall x \in \hat{X}$, then Γ is lhc at every interior point of X .*

Proof. Let \hat{x} be an interior point of X , $\hat{y} \in \Gamma(\hat{x})$, and $\langle x_n \rangle \subset X$ with $x_n \rightarrow \hat{x}$. Since x_n is convergent, choose $\varepsilon > 0$ such that $\hat{X} = B_\varepsilon(\hat{x}) \subset X$. Let D denote the boundary set of \hat{X} . We can represent x_n as a convex combination of \hat{x} and a point in D . That is, $\exists \alpha_n, d_n$ such that $x_n = \alpha_n d_n + (1 - \alpha_n)\hat{x}$ where $\alpha_n \in [0, 1]$ and $d_n \in D$. Since D is a bounded set, $\alpha_n \rightarrow 0$ as $x_n \rightarrow \hat{x}$. Choose \hat{Y} such that $\Gamma(x) \cap \hat{Y} \neq \emptyset, \forall x \in \hat{X}$. Then for each n , choose $\hat{y}_n \in \Gamma(d_n) \cap \hat{Y}$ so that $y_n = \alpha_n \hat{y}_n + (1 - \alpha_n)\hat{y}$. Since $(d_n, \hat{y}_n) \in A, \forall n, (\hat{x}, \hat{y}) \in A$, and A is convex, then $(x_n, y_n) \in A, \forall n$. Since $\alpha_n \rightarrow 0$ and $\hat{y}_n \in \hat{Y}, y_n \rightarrow \hat{y}$. Hence $\langle (x_n, y_n) \rangle \subset A$ and converges to (\hat{x}, \hat{y}) . ■

Theorem 284 *Let $\Gamma : X \rightarrow Y$ be a non-empty valued correspondence and let A be its graph. If (i) A is closed and (ii) for any bounded set $\widehat{X} \subset X$, the set $\Gamma(\widehat{X})$ is bounded, then Γ is compact valued and uhc.*

Proof. Compactness follows directly from (i) and (ii). Let $x_n \rightarrow \bar{x} \in X$ with $\langle x_n \rangle \subset X$. Since Γ is non-empty, $\exists y_n \in \Gamma(x_n), \forall n$. Since $x_n \rightarrow \bar{x}$, there is a bounded set $\widehat{X} \subset X$ such that $\langle x_n \rangle \subset \widehat{X}$ with $\bar{x} \in \widehat{X}$ by Theorem 164. Then by (ii), $\Gamma(\widehat{X})$ is bounded. Hence $\langle y_n \rangle \subset \Gamma(\widehat{X})$ has a convergent subsequence, $\langle y_{g(n)} \rangle \rightarrow \bar{y}$. Thus, $\langle (x_{g(n)}, y_{g(n)}) \rangle$ is a sequence in A converging to (\bar{x}, \bar{y}) . Since A is closed, $(\bar{x}, \bar{y}) \in A$. ■

In a future section we will use the following relationship between uhc of a correspondence and the closedness of its graph.

Theorem 285 *The graph of an uhc correspondence $\Gamma : X \rightarrow Y$ with closed values is closed.*

Proof. We have to prove that $X \times Y \setminus Gr(\Gamma)$ is open. Take $(x, y) \in X \times Y \setminus Gr(\Gamma)$ so that $y \notin \Gamma(x)$. Now we can choose an open neighborhood V_y of y in Y and $V_{\Gamma(x)}$ of $\Gamma(x)$ in Y such that $V_y \cap V_{\Gamma(x)} = \emptyset$. By (ii) of Theorem 280, $U_x = \Gamma^{+1}(V_{\Gamma(x)})$ is an open neighborhood of x in X , consequently $U_x \times V_y$ is an open neighborhood of (x, y) in $X \times Y$. Because $U_x \times V_y \cap Gr(\Gamma) = \emptyset$ we have $U_x \times V_y \subset X \times Y \setminus Gr(\Gamma)$ and hence $X \times Y \setminus Gr(\Gamma)$ is open. See Figure 4.7.7. ■

The converse of this theorem doesn't hold as the following example indicates.

Example 286 *Consider the function $F : \mathbb{R} \rightarrow \mathbb{R}$ given by*

$$F(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

F has a closed graph but is not uhc since it is clear that for an open set $(-\varepsilon, \varepsilon)$ in \mathbb{R} , $\Gamma^{+1}(-\varepsilon, \varepsilon) = (-\infty, -\frac{1}{\varepsilon}) \cup \{0\} \cup (\frac{1}{\varepsilon}, \infty)$, which is not open. However if the image $F(X)$ is compact, or a subset of a compact set, then the converse of Theorem 285 holds (i.e. a closed graph implies uhc). Hence, closedness of the graph can be used as a criterion of uhc. See Figure 4.7.8.

Theorem 287 *Let $\Gamma : X \rightarrow Y$ be a correspondence such that $\Gamma(X) \subset K$ where K is compact and the graph $Gr(\Gamma)$ is closed. Then Γ is uhc.*

Proof. Assume to the contrary that Γ is not uhc at x_0 . Then there exists an open neighborhood $V_{\Gamma(x_0)}$ of $\Gamma(x_0)$ in Y such that for every open neighborhood U_{x_0} of x_0 in X we have that $\Gamma(U_{x_0})$ is not contained in $V_{\Gamma(x_0)}$. We take $U_{x_0} = B_{\frac{1}{n}}(x_0)$, $n \in \mathbb{N}$. Then for every n we get a point $x_n \in B_{\frac{1}{n}}(x_0)$ such that $\Gamma(x_n)$ is not contained in $V_{\Gamma(x_0)}$. Let $y_n \in \Gamma(x_n)$ and $y_n \notin V_{\Gamma(x_0)}$. Then we have $\langle x_n \rangle \rightarrow x_0$ and $\langle y_n \rangle \subset K$. Since K is compact, there exists a subsequence $\langle y_{g(n)} \rangle \rightarrow y \in K$. Since $y_n \notin V_{\Gamma(x_0)}$, $\forall n$, this implies $y_n \in Y \setminus V_{\Gamma(x_0)}$. Since $Y \setminus V_{\Gamma(x_0)}$ is closed, then $y \in Y \setminus V_{\Gamma(x_0)}$ so that $y \notin V_{\Gamma(x_0)}$. Then we have $\langle x_n, y_{g(n)} \rangle \subset Gr(\Gamma)$ and $\langle x_n, y_{g(n)} \rangle \rightarrow (x_0, y)$. Since the $Gr(\Gamma)$ is closed, $(x_0, y) \in Gr(\Gamma)$. But this contradicts $y \notin V_{\Gamma(x_0)}$. ■

Now we state a few lemmas that will be very useful in the next chapter.

Definition 288 Let (X, d) be a metric space and $(Y, \|\cdot\|)$ be a normed vector space. Let $\Gamma : X \rightarrow Y$ be a correspondence. Then we can define two new correspondences: $\bar{\Gamma}$ (the closure of Γ) and $co(\Gamma)$ (the convex hull of Γ) by the following

$$\begin{aligned} \bar{\Gamma} & : X \rightarrow Y \text{ given by } \bar{\Gamma}(x) = \overline{\Gamma(x)}, \forall x \in X \\ co(\Gamma) & : X \rightarrow Y \text{ given by } (co(\Gamma))(x) = co\Gamma(x), \forall x \in X. \end{aligned}$$

Note $\bar{\Gamma}$ is by definition always closed valued and $co(\Gamma)$ is by definition always convex valued.

Example 289 $\Gamma : [0, 1] \rightarrow \mathbb{R}$ given by $\Gamma(x) = [0, x]$. Then $\bar{\Gamma}(x) = [0, x]$. See Figure 4.7.9.

Example 290 $\Gamma : [0, 1] \rightarrow \mathbb{R}$ given by $\Gamma(x) = \{0, 1\}$. Then $co(\Gamma(x)) = [0, 1]$. See Figure 4.7.10.

Lemma 291 If $\Gamma : X \rightarrow Y$ is lhc then $\bar{\Gamma}$ is also lhc.

Proof. The proof uses the following result:

$$\text{If } G \text{ is open in } Y \text{ and if } A \subset Y, \text{ then } A \cap G \neq \emptyset \text{ iff } \bar{A} \cap G \neq \emptyset. \quad (4.10)$$

Since $A \cap G \subset \bar{A} \cap G$, one direction is clear. Let $\bar{A} \cap G \neq \emptyset$. If $X \in \bar{A} \cap G$, then $X \in \bar{A}$ and if $X \in G$, then $\exists \langle x_n \rangle \rightarrow x$ and $x_n \in A$, $\forall n \in \mathbb{N}$. Since G is open, $\exists \varepsilon$ such that $B_\varepsilon(x) \subset G$. Since $\langle x_n \rangle \rightarrow x$, we have $x_n \in B_\varepsilon(x) \subset G$, $\forall n$ sufficiently large. Hence $x_n \in A \cap G$ so that $A \cap G \neq \emptyset$.

Now we need to prove that $\bar{\Gamma}^{-1}(V)$ is open in X if V is open in Y . But from (4.10), $\bar{\Gamma}^{-1}(V) = \Gamma^{-1}(V)$ which is open because Γ is lhc. ■

Lemma 292 *If Y is a normed vector space and $\Gamma : X \rightarrow Y$ is lhc, then $co(\Gamma)$ is lhc.*

Proof. Let $x \in X$, $\langle x_n \rangle \rightarrow x$, and $y \in co(\Gamma(x))$. We need to show that $\exists \langle y_n \rangle$ such that $\langle y_n \rangle \rightarrow y$ and $y_n \in co(\Gamma(x_n))$. Since $y \in co(\Gamma(x))$, then $y = \sum_{i=1}^m \lambda_i y^i$, where $y^i \in \Gamma(x)$ and $\sum_{i=1}^m \lambda_i = 1$. Since Γ is lhc, $\exists \langle y_n^i \rangle_{n=1}^{\infty}$ such that $y_n^i \in \Gamma(x_n)$ and $\langle y_n^i \rangle \rightarrow y^i$ for each $i = 1, \dots, m$. Let $y_n = \sum_{i=1}^m \lambda_i y_n^i$. Then $\langle y_n \rangle \rightarrow y$ and $y_n \in co(\Gamma(x_n))$. ■

Given two correspondences $\Gamma_1 : X \rightarrow Y$ and $\Gamma_2 : X \rightarrow Y$, provided that $\Gamma_1(x) \cap \Gamma_2(x) \neq \emptyset, \forall x \in X$, we can define a new correspondence

$$\Gamma_1 \cap \Gamma_2 : X \rightarrow Y$$

given by

$$(\Gamma_1 \cap \Gamma_2)(x) = \Gamma_1(x) \cap \Gamma_2(x)$$

Also let (X, d) be a metric space and $A \subset X$. The subset A can be expanded by a non-negative factor β denoted by $\beta + A$ where $\beta + A = \cup_{a \in A} B_{\beta}(a) = \{x \in X : d(x, A) < \beta\}$.²³ See Figure 4.7.11. Then for a correspondence $\Gamma : X \rightarrow Y$ where Y is a normed vector space, we have $\beta + \Gamma(x) = \{y \in Y : \|\Gamma(x) - y\| < \beta\}$. We say that $\beta + \Gamma(x)$ is a β -band around the set $\Gamma(x)$. See Figure 4.7.12.

We need the following lemma for Michael's selection theorem which is critical for the proof of a fixed point of a correspondence.

Lemma 293 *If Y is a normed vector space, if a correspondence F is defined by $F(x) = \beta + f(x)$ where f is a continuous function from X to Y , and if $\Gamma : X \rightarrow Y$ is a lhc correspondence, then $F \cap \Gamma$ is lhc.*²⁴

Proof. If $\langle x_n \rangle \rightarrow x$ and $y \in F(x) \cap \Gamma(x)$, then $y \in \Gamma(x)$. Since Γ is lhc, $\exists \langle y_n \rangle$ such that $y_n \in \Gamma(x_n)$ and $\langle y_n \rangle \rightarrow y$. We need to show that $y_n \in F(x_n)$ (i.e. $y_n \in (f(x_n) - \beta, f(x_n) + \beta)$) for n large enough. But by the

triangle property of a norm, we have

$$\|y_n - f(x_n)\| \leq \|y_n - y\| + \|y - f(x)\| + \|f(x) - f(x_n)\|. \quad (4.11)$$

²³Note that this distance between a point and a set is defined in 127.

²⁴Remember that F is a correspondence not a function; $F(x) = (f(x) - \beta, f(x) + \beta)$ which is an interval for every x .

The first term is sufficiently small because $\langle y_n \rangle \rightarrow y$ and the third term is sufficiently small because $\langle x_n \rangle \rightarrow x$ and f is continuous. Since $y \in F(x) = (f(x) - \beta, f(x) + \beta)$, the second term is less than β . Hence for n large enough, the right hand side of (4.11) is less than β and thus $y_n \in F(x_n)$. ■

4.7.1 Theorem of the Maximum

In economics, often we wish to solve optimization problems where households maximize their utility subject to constraints on their purchases of goods or firms maximize their profits subject to constraints given by their technology. In particular, consider the following example.

Example 294 *A household has preferences over two consumption goods (c_1, c_2) characterized by a utility function $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ given by $U(c_1, c_2) = c_1 + c_2$. The household has a positive endowment of good 2 denoted $\omega \in \mathbb{R}_+$. The household can trade its endowment on a competitive market to obtain good 1 where the price of good 1 in terms of good 2 is given by $p \in \mathbb{R}_+$. The household's purchases are constrained by its income; its budget set is given by*

$$B(p, \omega) = \{(c_1, c_2) \in \mathbb{R}_+^2 : pc_1 + c_2 \leq \omega\}.$$

Taking prices as given, the household's problem is

$$v(p, \omega) = \max_{(c_1, c_2) \in B(p, \omega)} U(c_1, c_2) \quad (4.12)$$

The first question we might ask is does a solution to this problem exist? When is it unique? How does it change as we vary parameters? The maximum theorem gives us an answer to these questions.

Before turning to the theorem, let us continue to work with Problem (4.12). First, let us establish properties of the budget set. In particular, we establish that if $p \in \mathbb{R}_{++}$, then $B(p, \omega)$ is a compact-valued, continuous correspondence. In this case, we will establish that the graph of the budget correspondence $A = \{(p, \omega, c_1, c_2) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : (c_1, c_2) \in B(p, \omega)\}$ satisfies the conditions of Theorems 283 and 284 only when $p > 0$. It is obviously non-empty since $(0, 0) \in B(p, \omega)$ for any $(p, \omega) \in \mathbb{R}_+^2$. The problem is that for any bounded set $\widehat{X} \subset \mathbb{R}_+^2$ of prices and incomes, there may not be a bounded set $\widehat{Y} \subset \mathbb{R}_+^2$ of consumptions. In particular, if $p > 0$, $B(p, \omega)$ is bounded since $0 \leq c_2 \leq \omega$ and $0 \leq c_1 \leq \frac{\omega}{p}$ but if $p = 0$, c_1 is unbounded.

See Figure 4.7.13. Under the assumption that $p > 0$, however, we have that $B(p, \omega)$ is a non-empty, compact valued, continuous correspondence.

Next we establish continuity of the utility function U . In particular, we show that $\forall \varepsilon > 0, \exists \delta > 0$ such that if

$$\sqrt{(c_1 - x_1)^2 + (c_2 - x_2)^2} < \delta, \quad (4.13)$$

then

$$|U(c_1, c_2) - U(x_1, x_2)| < \varepsilon. \quad (4.14)$$

Now rewrite the lhs of (4.14) as

$$|c_1 + c_2 - x_1 - x_2| \leq |c_1 - x_1| + |c_2 - x_2|$$

where the inequality follows from the triangle inequality. If we let $\delta = \frac{\varepsilon}{2}$, then (4.13) implies (4.14), establishing continuity of U . It is also instructive to graph the level sets (or “indifference curves”) of U . These are just given by the equations $c_2 = \bar{U} - c_1$ in Figure 4.7.14 as we vary \bar{U} . In the same figure we also plot budget sets with $p > 1$, $p = 1$, and $0 < p < 1$. It is simple to see from the figure that the solution, which we denote by “*”, to the household’s problem (4.12) is given by the demand correspondence

$$(c_1^*, c_2^*) = \begin{cases} (0, \omega) & \text{if } p > 1 \\ (x, \omega - x) \text{ with } x \in [0, \omega] & \text{if } p = 1 \\ (\frac{\omega}{p}, 0) & \text{if } p < 1 \end{cases} .$$

That is, if goods 1 and 2 are perfect substitutes for each other from the household’s preference perspective, then if good 1 is expensive (inexpensive), the household consumes none of (only) it, while if the two goods are the same price the possibilities are uncountable! Notice that the value function is continuous and increasing

$$v(p, \omega) = \begin{cases} \omega & \text{if } p \geq 1 \\ \frac{\omega}{p} & \text{if } 1 > p > 0 \end{cases} .$$

There is a more formal way of establishing the existence of a solution to such mathematical programming problems and how the solution varies with parameters. In general, let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $f : X \times Y \rightarrow \mathbb{R}$ be a single valued function, $\Gamma : X \rightarrow Y$ be a non-empty correspondence and consider the problem $\sup_{y \in \Gamma(x)} f(x, y)$. If for each x , $f(x, \cdot)$ is continuous in y and the

set $\Gamma(x)$ is compact, then we know from the Extreme Value Theorem 262 that for each x the maximum is attained. In this case,

$$v(x) = \max_{y \in \Gamma(x)} f(x, y) \quad (4.15)$$

is well defined and the set of values y which attain the maximum

$$G(x) = \{y \in \Gamma(x) : f(x, y) = v(x)\} \quad (4.16)$$

is non-empty (but possibly multivalued). The Maximum theorem puts further restrictions on Γ to ensure that v and G vary in a continuous way with x . The proof works in the following way. Consider a convergent sequence of elements in the constraint set $x_n \rightarrow x \in X$ (which we can always find since Γ is compact valued). By the extreme value theorem, there is a corresponding sequence of optimizing choices $y_n \in G(x_n)$ and $y_n \rightarrow y$. We must show that the limit of that sequence y is the optimizing choice in the constraint set defined at x . There are two parts to demonstrating this result. First we must show that y is in the constraint set ($y \in \Gamma(x)$). Then we must show y is the optimizing choice in $\Gamma(x)$.

Theorem 295 (Berge's Theorem of the Maximum) *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $f : X \times Y \rightarrow \mathbb{R}$ be a continuous function, and $\Gamma : X \rightarrow Y$ be a nonempty, compact-valued, continuous correspondence. Then $v : X \rightarrow \mathbb{R}$ defined in (4.15) is continuous and the correspondence $G : X \rightarrow Y$ defined in (4.16) is nonempty, compact valued, and uhc.*

Proof. The Extreme Value Theorem 262 ensures that for each x the maximum is attained and $G(x)$ is nonempty. Since $G(x) \subset \Gamma(x)$ and $\Gamma(x)$ is compact, $G(x)$ is bounded. To show $G(x)$ is closed, we suppose $y_n \rightarrow y$ with $y_n \in G(x)$, $\forall n$ and need to show that $y \in G(x)$.²⁵ Since $\Gamma(x)$ is closed, $y \in \Gamma(x)$. Since $v(x) = f(x, y_n) \forall n$ and f is continuous, then $v(x) = f(x, y)$ and $y \in G(x)$. Thus, $G(x)$ is nonempty and compact for each x .

To see that $G(x)$ is uhc, let $x_n \rightarrow x$ and choose $y_n \in G(x_n)$. We need to show that there exists a convergent subsequence $\langle y_{g(n)} \rangle \rightarrow y$ and $y \in G(x)$. Since Γ is uhc, $\exists \langle y_{g(n)} \rangle$ converging to $y \in \Gamma(x)$ by Theorem 280. Consider an alternative $z \in \Gamma(x)$. Since Γ is lhc, $\exists \langle z_{g(n)} \rangle$ converging to z with $z_{g(n)} \in \Gamma(x_{g(n)})$, $\forall g(n)$ by Theorem 278. Since $f(x_{g(n)}, y_{g(n)}) \geq$

²⁵This follows from Definition 111.

$f(x_{g(n)}, z_{g(n)})$, $\forall g(n)$ by optimality and f is continuous, $f(x, y) \geq f(x, z)$. Since this holds for any $z \in \Gamma(x)$, then $y \in G(x)$, satisfying uhc.

To see that $v(x)$ is continuous, fix x and let $x_n \rightarrow x$. Choose $y_n \in G(x_n)$, $\forall n$. Let $\bar{v} = \limsup v(x_n)$ and $\underline{v} = \liminf v(x_n)$. We can choose $\langle x_{g(n)} \rangle$ (a subsequence corresponding to $\langle y_{g(n)} \rangle$ above) such that $\bar{v} = \lim f(x_{g(n)}, y_{g(n)})$. Since G is uhc, $\exists \langle y_{h(g(n))} \rangle$ converging to $y \in G(x)$. Hence $\bar{v} = \lim f(x_{h(g(n))}, y_{h(g(n))}) = f(x, y) = v(x)$. An analogous argument establishes that $\underline{v} = v(x)$. Hence $\langle v(x_n) \rangle$ converges to $v(x)$. ■

The next three examples illustrate the Maximum theorem with simple mathematical problems.

Example 296 Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $f : Y \rightarrow \mathbb{R}$ be given by $f(y) = y$ and $\Gamma : X \rightrightarrows Y$ be given by

$$\Gamma(x) = \begin{cases} [0, 1] & \text{if } x \leq 1 \\ \frac{1}{2} & \text{if } x > 1 \end{cases}.$$

Consider the problem $v(x) = \max_{y \in \Gamma(x)} f(y)$. Then

$$v(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ \frac{1}{2} & \text{if } x > 1 \end{cases} \quad \text{and } G(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ \frac{1}{2} & \text{if } x > 1 \end{cases}.$$

Notice that $v(x)$ is not continuous and that $G(x)$ is not uhc. What condition of Theorem 295 did we violate? The constraint correspondence is not continuous; in particular, while $\Gamma(x)$ is uhc, it is not lhc. See Figure 4.7.15.

Example 297 Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $f : Y \rightarrow \mathbb{R}$ be given by $f(y) = \cos(y)$, and $\Gamma : X \rightrightarrows Y$ be given $\Gamma(x) = \{y \in Y : -x \leq y \leq x \text{ for } x \geq 0 \text{ and } x \leq y \leq -x \text{ for } x < 0\}$. Consider the problem $v(x) = \max_{y \in \Gamma(x)} f(y)$. Then

$$v(x) = 1, \forall x \text{ and } G(x) = \begin{cases} \{0\} & -2\pi < x < 2\pi \\ \{-2\pi, 0, 2\pi\} & -4\pi < x < 4\pi \\ \{-4\pi, -2\pi, 0, 2\pi, 4\pi\} & -6\pi < x < 6\pi \\ \text{etc} & \text{etc} \end{cases}.$$

Notice that $G(x)$ is uhc but not lhc since, for example, if we take $V = (2\pi - \varepsilon, 2\pi + \varepsilon)$ with $2\pi > \varepsilon > 0$, then $G(2\pi) \cap V \neq \emptyset$ but $\forall \delta > 0 \exists x' \in B_\delta(2\pi)$ such that $G(x') \cap V = \emptyset$ (in particular all those $x' < 2\pi$). See Figure 4.7.16.

Example 298 Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $f : Y \rightarrow \mathbb{R}$ be given by $f(y) = y^2$ and $\Gamma : X \rightarrow Y$ be given $\Gamma(x) = \{y \in Y : -x \leq y \leq x \text{ for } x \geq 0 \text{ and } x \leq y \leq -x \text{ for } x < 0\}$. Consider the problem $v(x) = \max_{y \in \Gamma(x)} f(y)$. Then

$$v(x) = x^2, \forall x \text{ and } G(x) = \{-x, x\}.$$

Notice that $G(x)$ is uhc and lhc but not convex valued. See Figure 4.7.17.

If we put more restrictions on the objective function and the constraint correspondence we can show that the set of maximizers $G(x)$ is single-valued and continuous.

Theorem 299 Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$. Let $\Gamma : X \rightarrow Y$ be a nonempty, compact- and convex- valued, continuous correspondence. Let A be the graph of Γ and assume $f : X \rightarrow \mathbb{R}$ is continuous function and that $f(x, \cdot)$ is strictly concave, for each $x \in X$.²⁶ If we define

$$g^*(x) = \arg \max_{y \in \Gamma(x)} f(x, y),$$

then $g^*(x)$ is a continuous function. If X is compact, then $g^*(x)$ is uniformly continuous.

Exercise 4.7.2 Prove Theorem 299.

We illustrate Theorem 299 through the next exercise.

Exercise 4.7.3 In Example 294 let the utility function $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be given by $u(c_1) + u(c_2)$ where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a strictly increasing, continuous, strictly concave function. Establish the following: (i) The objective function $U(c_1, c_2)$ is continuous and strictly concave on \mathbb{R}_+^2 ; (ii) The budget correspondence $B(p, y)$ is compact and convex; (iii) Existence and uniqueness of the set of maximizers; (iv) $v(p, y)$ is increasing in y and decreasing in p ; (v) $v(p, y)$ is continuous (try this as a proof by contradiction).

²⁶We say $f : \mathbb{R} \rightarrow \mathbb{R}$ is strictly concave if $f(\alpha x + (1 - \alpha)z) > \alpha f(x) + (1 - \alpha)f(z)$ for $x, z, \in \mathbb{R}$ and $\alpha \in [0, 1]$.

4.8 Fixed Points and Contraction Mappings

One way to prove the existence of an equilibrium of an economic environment amounts to showing there is a zero solution to a system of excess demand equations. In the case of Example 294, households take as given the relative price p and optimization may induce a continuous aggregate excess demand correspondence $ED(p)$ for good 1. If there is excess demand (supply), prices rise (fall) until equilibrium ($ED(\bar{p}) = 0$) is achieved. We may represent this “tatonnement” process by the mapping $f(p) = p + ED(p)$. In that case, an equilibrium is equivalent to a fixed point $\bar{p} = f(\bar{p})$.

Definition 300 *Let (X, d) be a metric space and $f : X \rightarrow X$ be a function or correspondence. We call $x \in X$ a **fixed point** of the function if $x = f(x)$ or of the correspondence if $x \in f(x)$.*

We now present four different fixed point theorems based upon different assumptions on the mapping f .

4.8.1 Fixed points of functions

The first fixed point theorem does not require continuity of f but uses only the fact that f is nondecreasing.

Theorem 301 (Tarsky) *Let $f : [a, b] \rightarrow [a, b]$ be a non-decreasing function (that is, if $x > y$ for $x, y \in [a, b]$, then $f(x) \geq f(y)$), $\forall a, b \in \mathbb{R}$ with $a < b$. Then f has a fixed point.*

Proof. Let $P = \{x \in [a, b] : x \leq f(x)\}$. We prove this in 4 parts. (i) Since $f(a) \in [a, b]$ implies $a \leq f(a)$, then $a \in P$ and hence P is non-empty. (ii) Since $P \subseteq [a, b]$ and $[a, b]$ is bounded, then P is bounded. Therefore, by the Completeness Axiom 3, $\bar{x} = \sup P$ exists. (iii) Since $\forall x \in P, x \leq \bar{x}$ by (ii), we have $f(x) \leq f(\bar{x})$ because f is nondecreasing. Since $x \in P, x \leq f(x) \leq f(\bar{x})$ so that $f(\bar{x})$ is an upper bound of P . Therefore, $\bar{x} \leq f(\bar{x})$ since \bar{x} is the least upper bound and hence $\bar{x} \in P$. (iv) Since $\bar{x} \leq f(\bar{x})$ implies $f(\bar{x}) \leq f[f(\bar{x})]$, we know $f(\bar{x}) \in P$. Therefore, $\bar{x} \geq f(\bar{x})$ since \bar{x} is an upper bound of P . Given that $\bar{x} \leq f(\bar{x})$ and $\bar{x} \geq f(\bar{x})$ we know that $\bar{x} = f(\bar{x})$. Note that we have not ruled out that there may be other points x' such that $x' = f(x')$. If so, then for all such points $x' \in P$. Our solution \bar{x} is the maximal fixed point. ■

The proof is illustrated in Figure 4.8.1. For a more general version of this proof, see Aliprantis and Border (1999, Theorem 1.8).

The next result by Brouwer requires that f be a continuous function. We saw a one dimensional version of it in section 4.6 which used the Intermediate value Theorem 255. That proof was very simple but the method we used there cannot be extended to higher dimensions. As it turns out proving it in \mathbb{R}^n where $n \geq 2$ is quite difficult. There are proofs that use calculus but we are going to present an elementary one based on simplexes which were introduced in section 4.5.2. Brouwer's fixed point theorem could be stated for a non-empty convex, compact subset of \mathbb{R}^n . Because a nondegenerate simplex is homeomorphic with (i.e. topologically equivalent to) a nonempty, convex, compact subset of \mathbb{R}^n it suffices to state Brouwer's theorem for the simplex.

Exercise 4.8.1 *Show that a simplex is homeomorphic with a nonempty, convex, compact subset $P = \{(p_1, p_2) \in \mathbb{R}^2 : 0 \leq p_1, p_2 \leq M, M \text{ finite}\}$.*

For notational simplicity and better intuition we prove Brouwer's theorem in \mathbb{R}^2 but this simplification has no effect whatsoever on the logic of the proof. The proof for general \mathbb{R}^n can be replicated with only minor notational changes.

Theorem 302 (Brouwer) *If $f(x)$ maps a nondegenerate simplex continuously into itself then there is a fixed point $x^* = f(x^*)$.*

Proof. (Sketch) The farther a point is from a vertex, the smaller is its barycentric coordinate. Thus, in Figure 4.8.12, a 's largest barycentric coordinate is the first one while b 's largest barycentric coordinate is the second one. For a given f , we introduce an indexing function $I(x)$ as follows. Let $y = f(x)$ and $I(x) = \min\{i : x_i > y_i\}$. If $b = f(a)$, then $I(a) = 0$ because $a_0 > b_0$. (the arrow connecting a with $f(a)$ points away from the vertex v_0).

x^* is a fixed point of f if $\alpha_i^* = \beta_i^*$, $i = 0, 1, 2$ where α_i^* and $\beta_i^* = f_i(x^*)$ are barycentric coordinates of x^* and $f(x^*)$. See Figure 4.8.4. In the case of barycentric coordinates, instead of equality (4.24) it suffices to show the following inequalities:

$$\alpha_i^* \geq \beta_i^*, i = 0, 1, 2 \tag{4.17}$$

because $\alpha_i^* \geq 0$, $\beta_i^* \geq 0$ and $\sum_{i=0}^2 \alpha_i^* = 1 = \sum_{i=0}^2 \beta_i^*$. Specifically, If f doesn't have a fixed point, then $I(x)$ is well defined for all $x \in S$ and obtains values

0, 1, or 2 with certain restrictions on the boundary. Divide the simplex into m^2 equal subsimplexes and index all the vertices of the the subsimplexes using $I(x)$ obeying restrictions on the boundary. Sperner's lemma guarantees that for each m there is at least one simplex with a complete set of indices (i.e. arrows originating at these vertices point inside the triangle). By choosing one vertex of such simplex for each m we get an infinite sequence of points from S that is the sequence is bounded. Hence by the Bolzano-Weierstrass theorem there exists a convergent subsequence with the limit point $x^* \in S$. As $m \rightarrow \infty$, a triangle collapses into one point (which is x^*) (at this point all arrows point inside itself). Since f is continuous, it preserves inequalities so that x^* is a fixed point of f . ■

In Chapter 6, we will introduce an infinite dimensional version of Brouwer's fixed point theorem by Schauder.

Example 303 (On Existence of Equilibrium) *Consider the following 2 period $t = 1, 2$ exchange problem with a large number I of identical agents. Let c_t^i, y_t^i, q_t denote an element (date t) of agent i 's consumption and endowment vector, as well as the price vector, respectively. Let a representative agent i 's budget set be given by $B(q, y^i) = \{c^i \in \mathbb{R}^2 : \sum_{t=1}^2 q_t c_t^i \leq \sum_{t=1}^2 q_t y_t^i\}$. Notice that an agent's budget set is homogeneous of degree zero in q .²⁷ That is, $B(\lambda q, y^i) = B(q, y^i)$. Thus, we are free to take $\lambda = \frac{1}{\sum_{t=1}^2 q_t} > 0$ and set $p = \left(\frac{q_1}{\sum_{t=1}^2 q_t}, \frac{q_2}{\sum_{t=1}^2 q_t} \right)$. This defines a one dimensional price simplex $S^1 = \{p \in \mathbb{R}_+^2 : \sum_{t=1}^2 p_t = 1\}$. Let the representative agent i 's utility function be given by $U(c^i) = \sum_{t=1}^2 \log(c_t^i)$. Exercise 4.7.3 establishes that $B(p, y^i)$ is a non-empty, compact- and convex-valued continuous correspondence and that $U(c^i)$ is strictly concave. Thus by version 299 of the Theorem of the Maximum the set of maximizers $\{c_1^i(p, y), c_2^i(p, y)\}$ are single valued and continuous functions. Since the sum of continuous functions is continuous, the aggregate excess demand function $z : S^1 \rightarrow \mathbb{R}^2$ given by $z(p) = \sum_{i=1}^I c^i(p, y) - y^i$ is continuous. It is a consequence of Walras Law that the inner product $\langle p, z(p) \rangle = 0$. To prove existence of equilibrium, we need to show that at the equilibrium price vector p^* there is no excess demand (i.e. $z(p^*) \leq 0$). Specifically, we must show that if $z : S^1 \rightarrow \mathbb{R}^2$ is continuous and satisfies $\langle p, z(p) \rangle = 0$, then $\exists p^* \in S^1$ such that $z(p^*) \leq 0$ (in the*

²⁷We say a function $f(x)$ is homogeneous of degree $k = 0, 1, 2, \dots$ if for any $\lambda > 0$, $f(\lambda x) = \lambda^k f(x)$.

case that all goods are desirable, this is $z(p^*) = 0$). To this end, define the mapping which raises the price of any good for which there is excess demand:

$$f_t(p) = \frac{p_t + \max(0, z_t(p))}{1 + \sum_{j=1}^2 \max(0, z_j(p))} \text{ for } t = 1, 2.$$

Notice that $f_t(p)$ is continuous since z_t and $\max(\cdot, \cdot)$ are continuous functions and that $f(p)$ lies in S^1 since $\sum_{t=1}^2 f_t(p) = 1$. By Brouwer's Fixed Point Theorem 302, there is a fixed point where $f(p^*) = p^*$. But this can be shown (by applying Walras Law) to imply that $z_t(p^*) \leq 0$. You should convince yourself that with these preferences and endowments, the markets for current and future goods are cleared if p^* implies $\frac{q_2}{q_1} = \frac{y_1}{y_2}$ so that the relative price of future goods in terms of current goods is lower the more plentiful future goods or less plentiful current goods are. Since $\frac{1}{1+r} = \frac{q_2}{q_1}$, this means that interest rates are higher the smaller is current output relative to future output. In other words, identical (representative) agents would like to borrow against plentiful future output to smooth consumption if current output is low; this would drive up the interest rate.

4.8.2 Contractions

Note that while the above theorems proved existence, they said nothing about uniqueness. The next set of conditions on f provide both.

Definition 304 Let (X, d) be a metric space and $f : X \rightarrow X$ be a function. Then f satisfies a **Lipschitz condition** if $\exists \gamma > 0$ such that $d(f(x), f(\tilde{x})) \leq \gamma d(x, \tilde{x})$, $\forall x, \tilde{x} \in X$. If $\gamma < 1$, then f is a **contraction mapping** (with **modulus** γ).

One way to interpret the Lipschitz condition is as a restriction on the slope of f . That is, $\frac{\Delta y}{\Delta x} = \frac{d(f(x), f(\tilde{x}))}{d(x, \tilde{x})} \leq \gamma$. Then a contraction is simply a function whose slope is everywhere less than 1. If f is Lipschitz, then it is uniformly continuous since we can take $\delta(\varepsilon) = \frac{\varepsilon}{\gamma}$ in which case $d(x, \tilde{x}) < \delta \Rightarrow d(f(x), f(\tilde{x})) < \varepsilon$. On the other hand, if f is uniformly continuous, it may not satisfy the Lipschitz condition as the next example shows.

Example 305 Let $f : [0, 1] \rightarrow \mathbb{R}$ be given by $f(x) = \sqrt{x}$. To see that f is uniformly continuous, for any $\varepsilon > 0$, let $\delta(\varepsilon) = \frac{\varepsilon^2}{2}$. Then if $|x - y| < \delta$, we have

$|\sqrt{x} - \sqrt{y}| \leq \sqrt{2|x-y|} < \sqrt{2 \cdot \frac{\varepsilon^2}{2}} = \varepsilon$ where the weak inequality follows since $(\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y \leq 2(\max\{x, y\} - \min\{x, y\}) = 2|x-y|$. To see that f is not Lipschitz, suppose so. Then for some $\gamma > 0$, $|\sqrt{x} - \sqrt{y}| \leq \gamma|x-y|$ or $\frac{|\sqrt{x} - \sqrt{y}|}{|x-y|} \leq \gamma$, $\forall x, y \in [0, 1]$. But $\frac{|\sqrt{x} - \sqrt{y}|}{|x-y|} = \frac{|\sqrt{x} - \sqrt{y}|}{|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|} = \frac{1}{|\sqrt{x} + \sqrt{y}|}$. Choose $x = \frac{1}{(1+\gamma)^2}$ and $y = 0$ so $x, y \in [0, 1]$. Then $\frac{1}{|\sqrt{x} + \sqrt{y}|} = 1 + \gamma$ which contradicts $\frac{|\sqrt{x} - \sqrt{y}|}{|x-y|} \leq \gamma$.

The next result establishes conditions under which there is a unique fixed point and provides a result on speed of convergence helpful for computational work.

Theorem 306 (Contraction Mapping) *If (X, d) is a complete metric space and $f : X \rightarrow X$ is a contraction with modulus γ , then f has a unique fixed point $\bar{x} \in X$ and (ii) for any $x_0 \in X$, $d(\bar{x}, f^n(x_0)) \leq \frac{\gamma^n}{1-\gamma}d(f(x_0), x_0)$ where f^n are iterates of f .²⁸*

Proof. Choose $x_0 \in X$ and define $\langle x_n \rangle_{n=0}^{\infty}$ by $x_{n+1} = f(x_n)$ so that $x_n = f^n(x_0)$. Since f is a contraction $d(x_2, x_1) = d(f(x_1), f(x_0)) \leq \gamma d(x_1, x_0)$. Continuing by induction,

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \gamma d(x_n, x_{n-1}) \leq \gamma^n d(x_1, x_0), n = 1, 2, \dots \quad (4.18)$$

For any $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \dots + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n) \quad (4.19) \\ &\leq [\gamma^{m-1} + \dots + \gamma^{n+1} + \gamma^n] d(x_1, x_0) \\ &= \gamma^n [\gamma^{m-n-1} + \dots + \gamma + 1] d(x_1, x_0) \\ &\leq \frac{\gamma^n}{1-\gamma} d(x_1, x_0) \end{aligned}$$

where the first line uses the triangle inequality and the second uses (4.18). It follows from (4.19) that $\langle x_n \rangle$ is a Cauchy sequence. Since X is complete, $x_n \rightarrow \bar{x}$. That \bar{x} is a fixed point follows since

$$\begin{aligned} d(f(\bar{x}), \bar{x}) &\leq d(f(\bar{x}), f^n(x_0)) + d(f^n(x_0), \bar{x}) \\ &\leq \gamma d(\bar{x}, f^{n-1}(x_0)) + d(f^n(x_0), \bar{x}) \end{aligned}$$

²⁸The iterates of f (mappings $\{f^n\}$), are defined as n -fold compositions $f^0(x) = x$, $f^1(x) = f(x)$, $f^2(x) = f(f^1(x))$, ..., $f^n(x) = f(f^{n-1}(x))$.

where the first line uses the triangle inequality and the second simply uses that f is a contraction. Since $\gamma < 1$, (4.18) implies $\lim_{n \rightarrow \infty} d(\bar{x}, f^{n-1}(x_0)) = 0 = \lim_{n \rightarrow \infty} d(f^n(x_0), \bar{x})$ so that $d(f(\bar{x}), \bar{x}) = 0$ or \bar{x} is a fixed point.

To prove uniqueness, suppose to the contrary there exists another fixed point x' . Then $d(x', \bar{x}) = d(f(x'), f(\bar{x})) \leq \gamma d(x', \bar{x})$ implies $\gamma \geq 1$, contrary to $\gamma < 1$ for a contraction.

Finally, the speed of convergence follows since

$$\begin{aligned} d(\bar{x}, f^n(x_0)) &\leq d(\bar{x}, f^m(x_0)) + d(f^m(x_0), f^n(x_0)) \\ &\leq \frac{\gamma^m}{1-\gamma} d(f(x_0), x_0) \end{aligned}$$

where the first line follows from the triangle inequality and the second from (4.19) and $\lim_{m \rightarrow \infty} d(\bar{x}, f^m(x_0)) = 0$. See Figure 4.8.13. ■

Sometimes it is useful to establish a unique fixed point on a given space X and then apply Theorem 306 again on a smaller space to characterize the fixed point more precisely.

Corollary 307 *Let (X, d) be a complete metric space and $f : X \rightarrow X$ be a contraction with fixed point $\bar{x} \in X$. If X' is a closed subset of X and $f(X') \subset X'$, then $\bar{x} \in X'$.*

Proof. Let $x_0 \in X'$. Then $\langle f^n(x_0) \rangle$ is a sequence in X' converging to \bar{x} . Since X' is closed, $\bar{x} \in X'$. ■

4.8.3 Fixed points of correspondences

In considering fixed points of correspondences we would like to utilize fixed point theorems (particularly Brouwer's fixed point theorem) of functions. How can we reduce multiple valued case to the single-valued one? This can be done by means of selection (i.e. a single-valued function that is *selected* from a multiple valued correspondence). Depending on circumstances we might have extra conditions on these choice functions. For instance, we might look for a continuous choice function (called continuous selection) or for a measurable choice function (called measurable selection, which we will deal with next chapter).

Definition 308 *Let $\Gamma : X \rightrightarrows Y$ be a correspondence, then the single-valued function $\Gamma_0 : X \rightarrow Y$ such that $\Gamma_0(x) \in \Gamma(x), \forall x \in X$ is called a **selection**. See Figure 4.8.14.*

The existence of a fixed point of a function proven by Brouwer requires continuity. Hence in this section we will deal with the problem of continuous selection. After proving the existence of a continuous selection, we will use Brouwer's fixed point theorem for functions to show that the selection has a fixed point, which is then obviously, a fixed point for the original correspondence. There are two main results of this subsection, Michael's continuous selection theorem and Kakutani's fixed point theorem for correspondences.

First we introduce a new notion, a partition of unity, that will be used in the proof of the selection theorem. Existence of a partition of unity is based on a well-known result from topology.

Lemma 309 (Urysohn) *Let A, B be two disjoint, closed subsets of a metric space X . Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0, \forall x \in A$ and $f(x) = 1, \forall x \in B$.*

For a proof, see Kelley (1957).

To continue we need to introduce the following topological concept.

Definition 310 *Let X be a metric space and let $\{G_i, i \in \Lambda\}$ be an open cover of X . Then a **partition of unity subordinate to the cover** $\{G_i\}$ is a family of continuous real-valued functions $\varphi_i : X \rightarrow [0, 1]$ such that $\varphi_i(x) = 0, \forall x \in X \setminus G_i$, and such that $\forall x \in X, \sum_{i \in \Lambda} \varphi_i(x) = 1$.*

Example 311 *Let $X = [0, 1], G_1 = [0, \frac{2}{3}], G_2 = (\frac{1}{3}, 1]$ be an open cover of $[0, 1]$. Let*

$$\varphi_1 = \left\{ \begin{array}{l} 1, 0 \leq x \leq \frac{1}{3} \\ -3(x - \frac{2}{3}), \frac{1}{3} < x \leq \frac{2}{3} \\ 0, \frac{2}{3} < x \leq 1 \end{array} \right\}$$

$$\varphi_2 = \left\{ \begin{array}{l} 0, 0 \leq x \leq \frac{1}{3} \\ 3(x - \frac{1}{3}), \frac{1}{3} < x \leq \frac{2}{3} \\ 1, \frac{2}{3} < x \leq 1 \end{array} \right\}$$

See Figure 4.8.15. Then $\{\varphi_1, \varphi_2\}$ is partition of unity subordinate to $\{G_1, G_2\}$.

Lemma 312 (Partition of unity) *Let X be a metric space and let $\{G_1, \dots, G_n\}$ be a finite open cover of X . Then there exists a partition of unity subordinate to this cover.*

Proof. We begin by constructing a new cover $\{H_1, \dots, H_n\}$ of X by open sets such that (i) $H_i \subset \overline{H}_i \subset G_i$ for $i = 1, 2, \dots, n$ and (ii) $\{H_i : i \leq j\} \cup \{G_i, i > j\}$ is a cover for each j . This is done inductively. Let $F_1 = X \setminus \bigcup_{i=2}^n G_i$. Then F_1 is closed and $F_1 \subset G_1$. The sets F_1 and $X \setminus G_1$ are closed disjoint subsets of the metric space X and hence can be separated by two disjoint open sets (see the separation axioms in Chapter 7), H_1 and $X - \overline{H}_1$. We have

$$F_1 \subset H_1 \subset \overline{H}_1 \subset G_1.$$

This satisfies (ii) for $j = 1$. Now suppose H_1, H_2, \dots, H_{k-1} have been constructed. Then since $\{H_i : i \leq k-1\} \cup \{G_i : i > k-1\}$ is a cover for X , $F_k = X \setminus ((\bigcup_{i=1}^{k-1} H_i) \cup (\bigcup_{i=k+1}^n G_i)) \subset G_k$. Again by separating F_k and $X \setminus G_k$ we get H_k such that $F_k \subset H_k \subset \overline{H}_k \subset G_k$. Clearly the collection $\{H_1, \dots, H_k\}$ satisfies (ii) with $j = k$. By Urysohn's lemma 309 we can construct real-valued functions ψ_i on X such that $\psi_i(x) = 0$ if $x \in X \setminus G_i$ and $\psi_i(x) = 1$ if $x \in H_i$ and $0 \leq \psi_i \leq 1$. Finally, let

$$\varphi_i(x) = \frac{\psi_i(x)}{(\sum_{j=1}^n \psi_j(x))}.$$

Since the collection $\{H_i, i = 1, 2, \dots, n\}$ is a cover, we have $\sum_{j=1}^n \psi_j(x) \neq 0$ for each x and hence $\varphi_i(x)$ is well-defined. $\{\varphi_i\}_{i=1}^n$ is the partition of unity subordinate to cover $\{G_i\}_{i=1}^n$. ■

Theorem 313 (Michael) *Let X be a metric space and Y be a Banach space, $\Gamma : X \rightarrow Y$ be lhc and $\Gamma(x)$ closed and convex for every $x \in X$. Then Γ admits a continuous selection.*

Proof. We prove the theorem under a stronger assumption, that X is a compact metric space, than is necessary.²⁹ We first show that for each positive real number β there exists a continuous function $f_\beta : X \rightarrow Y$ such that $f_\beta(x) \in \beta + \Gamma(x)$ for each $x \in X$. The desired selection will then be constructed as a limit of a suitable Cauchy sequence in such functions (that's why we need Y to be a complete normed vector space). For each $y \in Y$, let $U_y = \Gamma^{-1}(B_\beta(y))$ where $B_\beta(y)$ is an open ball around y of diameter β . Since Γ is lhc and $B_\beta(y)$ is open in Y , U_y is open in X . The collection

²⁹To prove the more general version we would need to use the concept of paracompactness which goes beyond the scope of this book. For the more general result, see Aubin-Frankowska (1990).

$\{U_y, y \in Y\}$ is then an open cover of X . Since X is compact there exists a finite subcollection $\{U_{y_i}, i = 1, \dots, n, y_i \in Y\}$ which by Lemma 312 (requiring a finite collection) has a partition of unity $\{\pi_i, i = 1, \dots, n\}$ subordinate to U_y . Let f_β be defined by $f_\beta(x) = \sum_{i=1}^n \pi_i(x) y_i$, where y_i is chosen in such a way that $\pi_i = 0$ in $X \setminus U_{y_i}$. Since $f_\beta(x)$ is the sum of finitely many continuous functions it is a continuous function from X to Y . f_β is a convex combination of those points y_i for which $\pi_i(x) \neq 0$. But $\pi_i(x) \neq 0$ only if $x \in U_{y_i}$. Thus $\Gamma(x) \cap B_\beta(y_i) \neq \emptyset$ and so $y_i \in \beta + \Gamma(x)$. Thus f_β is a convex combination of points y_i which lie in the convex set $\beta + \Gamma(x)$ and so f_β is also in that set (i.e. $f_\beta \in \beta + \Gamma(x)$).

Next we construct a sequence of such functions f_i to satisfy the following two conditions:

$$f_i(x) \in \frac{1}{2^{i-2}} + f_{i-1}(x), i = 2, 3, 4, \dots \quad (4.20)$$

$$f_i(x) \in \frac{1}{2^i} + \Gamma(x), i = 1, 2, 3, \dots \quad (4.21)$$

For f_1 we take the function f_β already constructed with $\beta = \frac{1}{2}$. Suppose that f_1, f_2, \dots, f_n have already been constructed. Let $\Gamma_{n+1}(x) = \Gamma(x) \cap (\frac{1}{2^n} + f_n(x))$. Then since $f_n(x)$ satisfies condition (4.21), $\Gamma_{n+1}(x)$ is non-empty and being the intersection of two convex sets is convex. Moreover $\Gamma_{n+1}(x)$ is lhc (see Lemma 293). Therefore by the first part of the proof and with $\beta = \frac{1}{2^{n+1}}$, there exists a function f_{n+1} with the property that $f_{n+1}(x) \in \frac{1}{2^{n+1}} + \Gamma_{n+1}(x)$. Since $\Gamma_{n+1}(x) \subset \Gamma(x)$, we have $f_{n+1}(x) \in \frac{1}{2^{n+1}} + \Gamma(x)$ so that condition (4.21) is satisfied. Furthermore, since $\Gamma_{n+1}(x) \subset \frac{1}{2^n} + f_n(x)$ we have $f_{n+1}(x) \in (\frac{1}{2^n} + \frac{1}{2^{n+1}}) + f_n(x) \subset \frac{1}{2^{n-1}} + f_n(x)$ which means that condition (4.20) is satisfied. We constructed the sequence $\langle f_i, i \in \mathbb{N} \rangle$ of functions for which $\|f_{n+1}(x) - f_n(x)\|_Y < \frac{1}{2^{n+1}}$ for all n and all x . Therefore $\sup_{x \in X} \|f_m(x) - f_n(x)\|_Y < \frac{1}{2^{n-2}}$ for all m, n with $m > n$. Thus the sequence $\langle f_i \rangle$ is a Cauchy sequence in the space of bounded continuous functions from X to Y which is complete because Y is complete (which we will see in Theorem 452 in Chapter 6). Then there exists a continuous function $f : X \rightarrow Y$ such that $\langle f_i \rangle \rightarrow f$ (with respect to the sup norm). Since (4.21) states that $\|f_n(x) - \Gamma(x)\| < \frac{1}{2^n}$ for all n , it follows that the limit function f has the property that $f(x) \in \bar{\Gamma}(x)$ (the closure of Γ). By assumption that $\Gamma(x)$ is closed we have $f(x) \in \Gamma(x)$ since $\bar{\Gamma}(x) = \Gamma(x)$. ■

Note that a correspondence that is uhc does not guarantee a continuous selection. See Example 272.

Combining Brouwer's fixed point theorem 302 with Michael's selection theorem 313 we immediately get the existence of a fixed point for lhc correspondences.

Corollary 314 *Let K be a non-empty, compact, convex subset of a finite dimensional space \mathbb{R}^m and let $\Gamma : K \rightarrow K$ be a lhc, closed, convex valued correspondence. Then Γ has a fixed point.*

Kakutani's theorem is usually stated with the condition that the correspondence Γ be uhc and closed valued. However, since we are dealing with a compact set K , by Theorems 285 and 287, uhc together with the closed valued property are equivalent to having a closed graph. It seems that this condition is somewhat easier to check.

In order to make the switch from uhc (or equivalently from closedness of graph) to lhc, we use the following lemma.

Lemma 315 *Let X and Y be compact subsets of a finite dimensional normed vector space \mathbb{R}^m and let $\Gamma : X \rightarrow Y$ be a convex-valued correspondence which has a closed graph (or equivalently is closed-valued and uhc). Then given $\beta > 0$, there exists a lhc, convex-valued correspondence $F : X \rightarrow Y$ such that $Gr(F) \subset \beta + Gr(\Gamma)$.*

Proof. Consider first the new correspondences \widehat{F}_ε defined for all $\varepsilon > 0$ by $\widehat{F}_\varepsilon(x) = \cup_{\widehat{x} \in X, \|x - \widehat{x}\| < \varepsilon} \Gamma(\widehat{x})$. To see that \widehat{F}_ε is lhc at x_0 , consider an open set G such that $\widehat{F}_\varepsilon(x_0) \cap G \neq \emptyset$. Then there exists $\widehat{x} \in X$ with $\|\widehat{x} - x_0\| < \varepsilon$ and $\Gamma(\widehat{x}) \cap G \neq \emptyset$. If μ is sufficiently small ($\mu < \varepsilon - \|\widehat{x} - x_0\|$) and if $\|x_0 - x\| < \mu$, then $\|\widehat{x} - x\| < \varepsilon$, and so $\widehat{F}_\varepsilon(x) \cap G \neq \emptyset$ because $\Gamma(\widehat{x}) \subset \widehat{F}_\varepsilon(x)$. Thus \widehat{F}_ε is lhc at an arbitrary $x_0 \in X$ and hence lhc on X . It follows from Lemma 292 that $F_\varepsilon = co(\widehat{F}_\varepsilon)$ is also lhc. Since F_ε is certainly convex-valued the proof is finished by showing that $Gr(F_\varepsilon) \subset \beta + Gr(\Gamma)$ if ε is sufficiently small. Suppose that it is not so. That is, for some $\beta > 0$ and all $n \in \mathbb{N}$, $Gr\left(F_{\frac{1}{n}}\right)$ is not contained in $\beta + Gr(\Gamma)$. Then there exists a sequence $\langle (x_n, y_n), n \in \mathbb{N} \rangle$ in $X \times Y$ such that $(x_n, y_n) \in Gr\left(F_{\frac{1}{n}}\right)$ but $d((x_n, y_n), Gr(\Gamma)) \geq \beta$. To say that $(x_n, y_n) \in Gr\left(F_{\frac{1}{n}}\right)$ means that $y_n = \sum_{i=1}^{m+1} \lambda_{n,i} y_{n,i}$ with $\lambda_{n,i} \geq 0$, $\sum_{i=1}^{m+1} \lambda_{n,i} = 1$, and $y_{n,i} \in \Gamma(\widehat{x}_{n,i})$ where $\|\widehat{x}_{n,i} - x_n\| < \frac{1}{n}$. Here we used Caratheodory's theorem 226 saying that in \mathbb{R}^m if y_n is a convex combination of certain points, it can always be expressed as a convex combination of

different ($m + 1$) points). Since X and Y are compact and $\lambda_{n,i} \in [0, 1]$ (which is also compact) all the above sequences contain subsequences (we will use the same indexes for subsequences) that converge, that is $\langle x_n \rangle \rightarrow x$, $\langle y_n \rangle \rightarrow y$, $\langle \lambda_{n,i} \rangle \rightarrow \lambda_i$ and $\langle \hat{x}_{n,i} \rangle \rightarrow \hat{x}_i$. Since $\|\hat{x}_{n,i} - x_n\| < \frac{1}{n}$, $\hat{x}_i = x$, for $i = 1, \dots, m+1$. We also have that $\sum_{i=1}^{m+1} \lambda_i = 1$ and $y_n = \sum_{i=1}^{m+1} \lambda_{n,i} y_{n,i} \rightarrow \sum_{i=1}^{m+1} \lambda_i y_i = y$. Now $(\hat{x}_{n,i}, y_{n,i}) \in Gr(\Gamma)$ and so $(\hat{x}_i, y_i) = (x, y_i) \in \overline{Gr(\Gamma)} = Gr(\Gamma)$ (since $Gr(\Gamma)$ is closed). Thus $y_i \in \Gamma(x)$ and, since $\Gamma(x)$ is convex, $y \in \Gamma(x)$ (being a convex combination of y_i). Hence $(x, y) \in Gr(\Gamma)$. But since $d((x_n, y_n), Gr(\Gamma)) \geq \beta$ for all n , this is not possible. This contradiction completes the proof. ■

Corollary 316 *In Lemma 315 we may also take F to be closed-valued.*

Proof. Let $F = \overline{F_\varepsilon}$ for sufficiently small ε . Then by lemma 291, F is lhc. It is of course still convex-valued and if $Gr(F_\varepsilon) \subset \frac{\beta}{2} + Gr(\Gamma)$, then $Gr(F) \subset \beta + Gr(\Gamma)$. ■

Theorem 317 (Kakutani) *Let K be a non-empty, compact, convex subset of finite-dimensional space \mathbb{R}^m and let $\Gamma : K \rightarrow K$ be a closed, convex valued, uhc correspondence (or convex valued with closed graph). Then Γ has a fixed point.*

Proof. By lemma 315 and corollary 316, for each $n \in \mathbb{N}$ there exists a lhc correspondence $F_n : K \rightarrow K$ such that $Gr(F_n) \subset \frac{1}{n} + Gr(\Gamma)$ and F_n has values which are closed and convex. Then by Michael's selection theorem 313, there is a continuous selection f_n for F_n . The function f_n is continuous mapping of K into itself and so, by Brouwer fixed point theorem 302, there exists $x_n \in K$ with $f_n(x_n) = x_n$. The compactness of K means that there exists a convergent subsequence of the sequence $\langle x_n \rangle$ such that $\langle x_{g(n)} \rangle \rightarrow x^*$. Since $(x_n, x_n) \in Gr(F_n) \subset \frac{1}{n} + Gr(\Gamma)$, it follows that $(x^*, x^*) \in \overline{Gr(\Gamma)} = Gr(\Gamma)$. Thus $x^* \in \Gamma(x^*)$ is a fixed point of Γ . ■

We now use an example to illustrate an important result due to Nash (1950). Nash's result says that *every* finite strategic form game has a mixed strategy equilibrium.

Example 318 *Reconsider the finite action coordination game in Example 282. We say that the mixed strategy profile (p^*, q^*) is a Nash Equilibrium*

if $\pi_1(p^*, q^*) \geq \pi_1(p, q^*)$ and $\pi_2(p^*, q^*) \geq \pi_2(p^*, q)$, $\forall p, q \in [0, 1]$. In Example 282, we showed

$$p^*(q) = \begin{cases} 0 & \text{if } q < \frac{1}{2} \\ [0, 1] & \text{if } q = \frac{1}{2} \\ 1 & \text{if } q > \frac{1}{2} \end{cases} \text{ and } q^*(p) = \begin{cases} 0 & \text{if } p < \frac{1}{2} \\ [0, 1] & \text{if } p = \frac{1}{2} \\ 1 & \text{if } p > \frac{1}{2} \end{cases}.$$

Given that the two agents are symmetric, to prove that the above game has a mixed strategy equilibrium, it is sufficient to show that $p^* : [0, 1] \rightarrow [0, 1]$ has a fixed point $p \in p^*(p)$. From Kakutani's theorem, it is sufficient to check that $p^*(p)$ is a non-empty, convex-valued, usc correspondence all of which was shown in Example 282. See Figure 4.8.16.

Exercise 4.8.2 Using Kakutani's theorem, prove Nash's result generally. See Fudenberg and Tirole p.29.

4.9 Appendix - Proofs in Chapter 4

Proof of Caratheodory Theorem 226. $x \in co(X)$ implies $x = \sum_{i=1}^m \lambda_i x_i$, $(x_1, \dots, x_m) \in X$, $\lambda_i > 0 \forall i$, and $\sum_{i=1}^m \lambda_i = 1$ by Theorem ?? . Suppose $m \geq n + 2$. Then the vectors

$$\begin{bmatrix} x_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} x_m \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$$

are linearly dependent. Hence there exist μ_1, \dots, μ_m , not all zero, such that

$$\sum_{i=1}^m \mu_i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = 0$$

(i.e. $\sum_{i=1}^m \mu_i x_i = 0$ and $\sum_{i=1}^m \mu_i = 0$). Let $\mu_j > 0$ for some j , $1 \leq j \leq m$. Define $\alpha = \frac{\lambda_j}{\mu_j} = \min \left\{ \frac{\lambda_i}{\mu_i} : \mu_i \neq 0 \right\}$ so that $\lambda_j - \alpha \mu_j = 0$. If we define

$\theta_i \equiv \lambda_i - \alpha \mu_i$, then $\theta_j = 0$, $\sum_{i=1}^m \theta_i = \sum_{i=1}^m \lambda_i - \alpha \sum_{i=1}^m \mu_i = 1 - \alpha \cdot 0 = 1$, and

$\sum_{i=1}^m \theta_i x_i = \sum_{i=1}^m \lambda_i x_i - \alpha \sum_{i=1}^m \mu_i x_i = \sum_{i=1}^m \lambda_i x_i = x$. Hence we expressed x as a

convex combination of $m - 1$ points of X with $\theta_j = 0$ for some j , reducing it from m points. If $m - 1 > n + 1$, then the process can be repeated until x is expressed as a convex combination of $n + 1$ points of X . ■

Proof of Theorem 248.. ($i \Rightarrow ii$) Let $a \in f^{-1}(V)$. Then $\exists y \in V$ such that $f(a) = y$. Since V is open, then $\exists \varepsilon > 0$ such that $B_\varepsilon(y) \subset V$. Since f is continuous for this ε , $\exists \delta(\varepsilon, a) > 0$ such that $\forall x \in X$ with $d_X(x, a) < \delta(\varepsilon, a)$ we have $d_Y(f(x), f(a)) < \varepsilon$. Hence $f(B_\delta(a)) \subset B_\varepsilon(f(a))$ or equivalently $B_\delta(a) \subset f^{-1}(B_\varepsilon(f(a))) \subset f^{-1}(V)$.

($ii \Rightarrow iii$) Let $\langle x_n \rangle \rightarrow x$. Take any open ε -ball $B_\varepsilon(f(x)) \subset V$. Then $x \in f^{-1}(B_\varepsilon(f(x)))$ and $f^{-1}(B_\varepsilon(f(x)))$ is open (by assumption ii). Now $\exists \delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$. Since $\langle x_n \rangle \rightarrow x$, $\exists N$ such that $n \geq N$, $x_n \in B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$. Hence $f(x_n) \in B_\varepsilon(f(x)) \forall n \geq N$ so $f(x_n) \rightarrow f(x)$. See Figure 4.6.5.

($iii \Rightarrow i$) It is sufficient to prove the contrapositive. Thus, suppose $\exists \varepsilon > 0$ such that $\forall \delta = \frac{1}{n}$, $\exists x_n$ such that $d_X(x_n, x) < \frac{1}{n}$ and $d_Y(f(x_n), f(x)) \geq \varepsilon$. Thus we have a sequence $\langle x_n \rangle \rightarrow x$ but none of the elements of $\langle f(x_n) \rangle$ is in an ε -ball around $f(x)$. Hence $\langle f(x_n) \rangle$ doesn't converge to $f(x)$.³⁰

■

Proof of Theorem 268.. In the first step, for a given $\varepsilon > 0$, we construct δ that depends only on ε . Thus, take $\varepsilon > 0$. Since f is continuous on X , then for any $x \in X$ there is a number $\delta(\frac{1}{2}\varepsilon, x) > 0$ such that if $x' \in X$ and $d(x, x') < \delta(\frac{1}{2}\varepsilon, x)$, then $d_Y(f(x), f(x')) < \frac{1}{2}\varepsilon$. The collection of open balls $\mathcal{G} = \{B_{\delta(\frac{1}{2}\varepsilon, x)}(x), x \in X\}$ is an open covering of X . Since X is compact there exists a finite subcollection, say $\{B(x_1), \dots, B(x_n)\}$ of these balls that covers X . Then define $\delta(\varepsilon) = \frac{1}{2} \min\{\delta(\frac{1}{2}\varepsilon, x_1), \dots, \delta(\frac{1}{2}\varepsilon, x_n)\}$ which is obviously independent of x .

In the second step, we use the $\delta(\varepsilon)$ constructed above to establish uniform continuity. Suppose that $x, x' \in X$ and $d_X(x', x) < \delta(\varepsilon)$. Because $\{B(x_1), \dots, B(x_n)\}$ covers X , $x \in B(x_k)$ for some k . That is

$$d_X(x, x_k) < \frac{1}{2}\delta(\frac{1}{2}\varepsilon, x_k). \quad (4.22)$$

By the triangle inequality it follows that

$$d_X(x', x_k) \leq d_X(x', x) + d_X(x, x_k) \leq 2\delta(\varepsilon) \leq \delta(\frac{1}{2}\varepsilon, x_k). \quad (4.23)$$

³⁰Munkres p.127 Th10.1, sequences see Munkres p. 128, Th 10.3.

Then (4.22) and continuity of f at x_k imply $d_Y(f(x)), f(x_k)) < \frac{1}{2}\varepsilon$, while (4.23) and continuity of f at x_k imply $d_Y(f(x')), f(x_k)) < \frac{1}{2}\varepsilon$. Again by the triangle inequality it follows that

$$d_Y(f(x), f(x')) \leq d_Y(f(x), f(x_k)) + d_Y(f(x'), f(x_k)) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Thus, we have shown that if $x, x' \in X$ for which $d_X(x', x) < \delta(\varepsilon)$, then $d_Y(f(x), f(x')) < \varepsilon$. ■

Proof of Brouwer's Fixed Point Theorem 302. (in \mathbb{R}^2). Let $f : S \rightarrow S$ be continuous, where S is a fixed nondegenerate simplex with vertices v^0, v^1, v^2 . $x^* = f(x^*)$ implies that

$$\alpha_i^* = \beta_i^*, i = 0, 1, 2 \tag{4.24}$$

where α_i^* and $\beta_i^* = f_i(x^*)$ are barycentric coordinates of x^* and $f(x^*)$. See Figure 4.8.4. In the case of barycentric coordinates, instead of equality (4.24) it suffices to show the following inequalities.

$$\alpha_i^* \geq \beta_i^*, i = 0, 1, 2 \tag{4.25}$$

(4.24) and (4.25) are equivalent because $\alpha_i^* \geq 0, \beta_i^* \geq 0$ and $\sum_{i=0}^2 \alpha_i^* = 1 = \sum_{i=0}^2 \beta_i^*$. To see this, note that

$$\begin{aligned} \alpha_0 &\geq \beta_0 \geq 0, \\ \alpha_1 &\geq \beta_1 \geq 0, \\ \alpha_2 &\geq \beta_2 \geq 0, \\ \alpha_0 + \alpha_1 + \alpha_2 &= \beta_0 + \beta_1 + \beta_2 (= 1), \\ \text{and } (\alpha_0 - \beta_0) + (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) &= 0 \\ \text{implies } \alpha_0 &= \beta_0, \alpha_1 = \beta_1, \alpha_2 = \beta_2. \end{aligned}$$

Let $y = f(x)$. If $y \neq x$, then some coordinates $\beta_i \neq \alpha_i$. Therefore, since $\sum \alpha_i = \sum \beta_i$ we must have both some $\alpha_k > \beta_k$ and some $\alpha_i < \beta_i$. Focus on the first inequality, $\alpha_k > \beta_k$; this cannot occur when $\alpha_k = 0$. In other words this cannot occur on the boundary segment opposite vertex v^k (see the calculations in Example 228). For example, on the boundary line segment joining v^0 and v^1 opposite v^2 (we will denote this line segment $\underline{(v^0, v^1)}$), the inequality $\alpha_2 > \beta_2$ cannot occur because $\alpha_2 = 0$ for all these points but $0 > \beta_2 \geq 0$ is false. See Figure 4.8.5.

Now we introduce an indexing scheme for points in the simplex as follows. Given functions $y = f(x)$ ($f : S \rightarrow S$), for each $x \in S$ such that $x \neq y = f(x)$ we have seen that $x_i > y_i$ for some i . Now define $\mathcal{I}(x)$ as the smallest such i (that is, $\mathcal{I}(x) = \min \{i : \alpha_i > \beta_i\}$). Hence $\mathcal{I}(x)$ can obtain values 0, 1, 2 (in our case for \mathbb{R}^2). These values depend on the function $y = f(x)$ of course but on the boundary, we know that $\mathcal{I}(x)$ is restricted. For example on the boundary $\overline{(v^0, v^1)}$ where $\alpha_2 = 0$ we can't have $\alpha_2 > \beta_2$ so that $\mathcal{I}(x)$ can't obtain 2. Thus $\mathcal{I}(x) = 0$ or 1 on the line segment $\overline{(v^0, v^1)}$. In general, $\mathcal{I}(x)$ satisfies the same set of restrictions as $I(x)$ in (4.3) in Section 4.5.2 and hence we can use the results of Sperner's Lemma 229.

Why are we doing this? We are looking for a fixed point of $y = f(x)$. That is a point x whose barycentric coordinates satisfy all the inequalities $\alpha_0 \geq \beta_0$, $\alpha_1 \geq \beta_1$, $\alpha_2 \geq \beta_2$. To do so, for $m = 2, 3, 4, \dots$, we form the m th barycentric subdivision of our simplex S . For example, see Figure 4.8.7 for $m = 2$. The vertices in the subdivision are points $z = \frac{1}{2}(\mu_0, \mu_1, \mu_2)$ where the μ_i are integers (and $\frac{\mu_i}{2}$, $i = 0, 1, 2$ are barycentric coordinates with respect to the m th subdivision) with all $\mu_i \geq 0$ and $\sum_{j=0}^2 \mu_j = 2$. In general for the m -th subdivision, the vertices are the points $x = \frac{1}{m}(\mu_0, \mu_1, \mu_2)$. Where μ_i are integers satisfying all $\mu_i \geq 0$, $\sum_{i=0}^2 \mu_i = m$. We will call a little shaded triangle a cell. The original simplex is the whole body. For $m = 5$ we see 25 cells in Figure 4.8.8. Each cell is small; the diameter of each cell is $\frac{1}{5}$ of the diameter of the body. In general, in the m -th subdivision of a simplex, the number of cells m^2 tends to infinity as $m \rightarrow \infty$ and the diameter of each cell tends to zero. If Δ is the diameter of the body, then the diameter of each cell is $\frac{\Delta}{m}$.

We are given a continuous function $y = f(x)$ that maps the simplex into itself. We assume that $f(x)$ has no fixed point and we show that this leads to contradiction. Since we assume $x \neq y = f(x)$ (i.e. no fixed point) we may use the indexing function $\mathcal{I}(x)$ for each point $x \in S$. The index takes one of the values 0, 1, 2 at each point of the body and on the boundary of the simplex. The index satisfies the restrictions (??). For example in Figure 4.8.9 there are 21 vertices. Label each vertex x with an index $\mathcal{I}(x) = 0, 1, 2$ arbitrarily except that this indexing has to obey the restrictions (??) on the boundary. That means you must use $\mathcal{I} = 0$ or 1 on the bottom side, $\mathcal{I} = 0$ or 2 on the left and $\mathcal{I} = 1$ or 2 on the right. Also $\mathcal{I} = 0$ at v^0 , $\mathcal{I} = 1$ at v^1 and $\mathcal{I} = 2$ at v^2 . This leaves 6 interior vertices, each to be labeled arbitrarily 0, 1, or 2. Try to label these vertices such that none of the 25 cells has all the

labels 0, 1, 2. No matter how hard you try, at least one of the cells must have a complete set of labels. This is guaranteed by Sperner's Lemma 229 which follows immediately after this proof. In particular, the lemma guarantees that for any m , in the m -th subdivision there is a cell with a complete set of labels, say

$$\begin{aligned}\mathcal{I} &= 0 \text{ at the vertex } x^0(m) \\ \mathcal{I} &= 1 \text{ at the vertex } x^1(m) \\ \mathcal{I} &= 2 \text{ at the vertex } x^2(m)\end{aligned}\tag{4.26}$$

What does this mean for the function $y = f(x)$? If $I = j$ then for barycentric coordinates of the points x and y we have $\alpha_j > \beta_j$. Therefore, (4.26) implies

$$\begin{aligned}\alpha_0 &> \beta_0 \text{ at } x^0(m) \\ \alpha_1 &> \beta_1 \text{ at } x^1(m) \\ \alpha_2 &> \beta_2 \text{ at } x^2(m)\end{aligned}\tag{4.27}$$

If m is large all the vertices of the cell are close to each other, since the diameter of the cell is $\frac{\Delta}{m}$. Therefore

$$\max_{0 \leq i < j \leq 2} |x^i(m) - x^j(m)| = \frac{\Delta}{m} \longrightarrow 0 \text{ as } m \longrightarrow \infty.\tag{4.28}$$

As $m \rightarrow \infty$, what can be said about the vertices (say $x^0(m)$)? This vertex might move unpredictably through the simplex in some bounded infinite sequence. See Figure 4.8.10. Since S is compact, by the Bolzano-Weierstrass Theorem 180 this sequence contains a subsequence that has a limit, say $x^0(m_s) \rightarrow x^*$ as $s \rightarrow \infty$. The limit point $x^* \in S$ because S is closed.

But because of the closeness of the vertices, (4.28) implies that all tend to x^* as $m_s \rightarrow \infty$. $x^p(m_s) \rightarrow x^*$ as $s \rightarrow \infty$, $p = 0, 1, 2$. Now the continuity of $f(x)$ implies $f(x^p(m_s)) \rightarrow f(x^*) = y^*$ as $s \rightarrow \infty$, $p = 0, 1, 2$. But the barycentric coordinates of a point x depend continuously on x . Therefore, if we let $m = m_s \rightarrow \infty$ in (4.27) we obtain the limiting inequalities

$$\begin{aligned}\alpha_0 &\geq \beta_0 \text{ at the limit } x^* \iff \alpha_0^* \geq \beta_0^* = f_0(x^*) \\ \alpha_1 &\geq \beta_1 \text{ at the limit } x^* \iff \alpha_1^* \geq \beta_1^* = f_1(x^*) \\ \alpha_2 &\geq \beta_2 \text{ at the limit } x^* \iff \alpha_2^* \geq \beta_2^* = f_2(x^*)\end{aligned}$$

But we know by (4.25) that these inequalities imply equalities, thus $x^* = y^* = f(x^*)$. ■

Proof. Figures for Sections ?? to 4.8

Figure ??.1: Open Sets ■

Figure ??.2: Sup Balls and Open Neighborhoods

Figure ??.3: $(0, 1)$ vs $(0, 1]$ Figure ??.4: $\{(x, y) | 0 < x < 1, y = 2\}$

Figure ??.5: Closure and Boundary Points

Figure 4.1.1: On Cluster points and the Limit of $\langle (-1)^n \rangle$ Figure 4.1.2: On the Limit of $\langle \left(\frac{1}{n}\right) \rangle$ Figure 4.1.3: On the Limit of $\langle \left(\frac{x}{n}\right) \rangle$ Figure 4.1.4: On the Limit of $\langle x^n \rangle$ Figure 4.3.1: Construction of $(\Rightarrow)H$ closed.

Figure 4.3.2: Compactness for General Metric Spaces.

Figure 4.4.1: A disconnected set

Figure 4.6.1: Pointwise continuity in \mathbb{R}

See Figure 4.7.1: Lower Hemicontinuity

See Figure 4.7.2: Upper Hemicontinuity

See Figure 4.7.3: Best Response Correspondence

See Figure 4.7.4: Budget Sets with $p = 0$ and $p > 0$

See Figure 4.7.5: Demand Correspondence with Linear Preferences

Figure 4.8.1: Tarski's Fixed Point Theorem in $[a, b]$ Figure 4.8.2: Brouwer's Fixed Point Theorem in $[a, b]$ Figure 4.8.3: Fixed Point of a Contraction Mapping in $[a, b]$ Figure 4.8.4: Kakutani's Fixed Point Theorem in $[a, b]$

Figure 4.8.5: Existence of Nash Equilibria

Figure 4.5.1: Open Sets

4.10 Bibliography for Chapter 4

Sections ?? to are based on Royden (Chapters 2 and 7) and Bartle (Sections 9,14-16). Section 4.2 is based on Royden (Chapter 7, Section 4) and Munkres (Chapter 7, Section 1). Section 4.3 is based on Munkres (Chapter 3, Sections 5 and 7, Chapter 7, Section 3), Royden (Chapter 7, Section 7), and Bartle (Chapter 11). Section 4.6 is based on Munkres (Chapter 3,). Section 4.5 is from Bartle (Sec 8).

4.11 End of Chapter Problems

1) The next results (from Definition 319 to Theorem 164) require a total ordering of a set X , so we restrict X to be \mathbb{R} .

Definition 319 Let $\langle x_n \rangle$ be a bounded sequence in \mathbb{R} . The **limit superior** of $\langle x_n \rangle$, denoted $\limsup x_n$ or $\overline{\lim} x_n$, is given by $\inf_n \sup_{k \geq n} x_k$. The **limit inferior** of $\langle x_n \rangle$, denoted $\liminf x_n$ or $\underline{\lim} x_n$, is given by $\sup_n \inf_{k \geq n} x_k$.

That is, $l \in \mathbb{R}$ is the limit superior of $\langle x_n \rangle$ iff given $\varepsilon > 0$, there are at most a finite number $n \in \mathbb{N}$ such that $l + \varepsilon < x_n$ but there are an infinite number such that $l - \varepsilon < x_n$. The limit superior is just the maximum cluster point and the limit inferior is just the minimum cluster point.

Example 320 Recall Example 141 where we considered the sequence $\langle (-1)^n \rangle$ which had two cluster points. There, $\liminf x_n = -1$ and $\limsup x_n = 1$. To see why the limit inferior is -1 , consider: $n = 1$ has $\inf_{k \geq 1} x_k = -1$, $n = 2$ has $\inf_{k \geq 2} x_k = -1$; and any given n has $\inf_{k \geq n} x_k = -1$. But then the $\sup\{-1, -1, \dots\}$ is just -1 .

Theorem 321 Let $\langle x_n \rangle$ be a bounded sequence of real numbers. Then $\lim x_n$ exists iff $\liminf x_n = \limsup x_n = \lim x_n$.

Exercise 4.11.1 Prove Theorem 321.

While Theorem 321 hinges on the fact that \mathbb{R} is totally ordered, a similar result holds for any totally ordered set.

Example 322 Recall Example 137 where we considered the sequence $\langle \frac{1}{n} \rangle$. It is simple to see that it has a cluster point at 0 since any open ball around 0 of size δ has an infinite number of elements in the sequence past $N(\delta) = \lfloor \frac{1}{\delta} \rfloor + 1$ contained in it. Furthermore, $\liminf x_n = 0 = \limsup x_n$. To see why the limit superior is 0, consider: $n = 1$ has $\sup_{k \geq 1} x_k = 1$, $n = 2$ has $\sup_{k \geq 2} x_k = \frac{1}{2}$; and any given n has $\sup_{k \geq n} x_k = \frac{1}{n}$. But then the $\inf\{1, \frac{1}{2}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots\}$ is just 0.

Example 323 Consider $\langle (-1)^n + \frac{1}{n} \rangle_{n \in \mathbb{N}}$. Then $\langle x_n \rangle = \langle -2, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}, \dots \rangle$. See Figure 4.1.5. The cluster points of $\langle x_n \rangle$ are -1 and 1 , which are also the limit inferior and limit superior, respectively. Notice that the subsequence of odd numbered indices $\langle x_{2k-1} \rangle = \langle (-1)^{2k-1} + \frac{1}{2k-1} \rangle_{k=1}^{\infty} = \langle -2, -\frac{2}{3}, -\frac{4}{5}, \dots \rangle \rightarrow -1$ and the subsequence of even numbered indices $\langle x_{2k} \rangle = \langle (-1)^{2k} + \frac{1}{2k} \rangle_{k=1}^{\infty} = \langle \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \dots \rangle \rightarrow 1$.

Note that while a limit point is unique, we saw in Example 141 that a sequence can have many cluster points. In that case, the smallest cluster point is called the limit inferior and the largest cluster point is called the limit superior.

2) We provide another useful criterion in \mathbb{R} to establish convergence, which is true only because \mathbb{R} is totally ordered and complete.

Theorem 324 (Monotone Convergence) Let $\langle x_n \rangle$ be a monotone increasing sequence (i.e. $x_1 \leq x_2 \leq \dots \leq x_i \leq x_{i+1} \leq \dots$) in the metric space $(\mathbb{R}, |\cdot|)$.³¹ Then $\langle x_n \rangle$ converges iff it is bounded and its limit is given by $\lim x_n = \sup\{x_n | n \in \mathbb{N}\}$.

Proof. (\Rightarrow) Boundedness follows by Lemma 164, so all we must show is $\bar{x} = \sup\{x_n\}$. Convergence implies $\bar{x} - \delta < x_n < \bar{x} + \delta$, $\forall n \geq N(\delta)$ by definition 136. As a property of the supremum, we know that if $x_n < y_n$, $\forall n \in \mathbb{N}$, then $\sup x_n \leq \sup y_n$, $\forall n \in \mathbb{N}$. This implies $\bar{x} - \delta \leq \sup\{x_n\} \leq \bar{x} + \delta$ or $|\sup\{x_n\} - \bar{x}| \leq \delta$.

(\Leftarrow) If $\langle x_n \rangle$ is a bounded, monotone increasing sequence of real numbers, then by the Completeness Axiom 3.3 its supremum exists (call it $x' = \sup\{x_n\}$). Since x' is a sup, $x' - \delta$ is not an ub and $\exists K(\delta) \in \mathbb{N}$ such that $x' - \delta < x_{K(\delta)}$ for any $\delta > 0$.³² Since $\langle x_n \rangle$ is monotone, $x' - \delta < x_n \leq x' < x' + \delta$, $\forall n \geq K(\delta)$ or $|x_n - x'| < \delta$. ■

Example 325 Re-consider Example 137 where $\langle \frac{1}{n} \rangle_{n \in \mathbb{N}}$. It is clear that this sequence is monotone decreasing with infimum 0 , which is also its limit.

Exercise 4.11.2 Consider the sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ given by $\langle (1 + \frac{1}{n})^n \rangle_{n \in \mathbb{N}}$. Show that this sequence is increasing and bounded above so that by the Monotone Convergence Theorem 324, the sequence converges in $(\mathbb{R}, |\cdot|)$.

³¹That is, $x_1 \leq x_2 \leq \dots \leq x_i \leq x_{i+1} \leq \dots$

³²Existence of this index follows from property (ii) in the footnote to definition ?? of a supremum.

3)

Exercise 4.11.3 *Let (X, d) be totally bounded. Show that X is separable.*

Chapter 5

Measure Spaces

Many problems in economics lend themselves to analysis in function spaces. For example, in dynamic programming we define an operator that maps functions to functions. As in the case of metric spaces, we need some way to measure distance between the elements in the function space. Since function spaces are defined on uncountably infinite dimensional sets, the distance measure involves integration.¹ In this chapter we will focus primarily on Lebesgue integration. Since Lebesgue integration can be applied to a more general class of functions than the more standard Riemann approach, this will allow us to consider, for example, successive approximations to a broader class of functional equations in dynamic programming.

To understand Lebesgue integration we focus on measure spaces. This has the added benefit of introducing us to the building blocks of probability theory. In probability theory, we start with a given underlying set X and assign a probability (just a real valued function) to subsets of X . For instance, if the experiment is a coin toss, then $X = \{H, T\}$ and the set of all possible subsets is given by $\mathcal{P}(X) = \{\emptyset, \{H\}, \{T\}, X\}$ described in Definition 9. Then we assign zero probability to the event where the flip of the coin results in neither an H nor a T (i.e. $\mu(\emptyset) = 0$), we assign probability one to the event where the flip results in either H or T (i.e. $\mu(X) = 1$), and we assign probability $\frac{1}{2}$ to the event where the flip of the fair coin results in H (i.e. $\mu(H) = \frac{1}{2}$).

¹In Section 4.5, we saw that in the (ℓ_p) space of (countably) infinite sequences, the distance measure involved countable sums; that is, $d(\langle x_n \rangle, \langle y_n \rangle) = (\sum_{n=1}^{\infty} (x_n - y_n)^p)^{\frac{1}{p}}$. Integration is just the uncountable analogue of summation.

One of the important results we show in this Chapter is that the collection of Lebesgue measurable sets is a σ -algebra in Theorem 341 and that the collection of Borel sets is a subset of the Lebesgue measurable sets in Theorem 346. Then we introduce the concept of measurability of a function and a correspondence and define the Lebesgue integral of measurable functions. Then we provide a set of convergence theorems for the existence of a Lebesgue integral which are applicable under different conditions. These are the Bounded Convergence Theorem 386, Fatou's Lemma 393, the Monotone Convergence Theorem 396, the Lebesgue Dominated Convergence Theorem 404, and Levi's Theorem 407. Essentially these provide conditions under which a limit can be interchanged with an integral. Then we introduce general and signed measures. Here we have two important results, namely the Hahn Decomposition Theorem 427 of a measurable space with respect to a signed measure and the Radon-Nikodym Theorem 434 where a signed measure can be represented simply by an integral. The chapter is concluded by introducing an example of a function space (which is the subject of the next chapter 6). In particular, we focus on the space of integrable functions, denoted L_1 , and prove it is complete in Theorem 443.

5.1 Lebesgue Measure

Before embarking on the general definition of a measure space, in the context of a simple set $X = \mathbb{R}$ we will introduce the notion of length (again just a real-valued function defined on a subset of \mathbb{R}), describe desirable properties of a measure space, and describe a simple measure related to length.

Definition 326 *A **set function** associates an extended real number to each set in some collection of sets. In \mathbb{R} , the **length** $l(I)$ of an interval $I \subset \mathbb{R}$ is the difference of the endpoints of I .² Thus, in the case of the set function length, the domain is the collection of all intervals.*

We would like to extend the notion of length to more complicated sets than intervals. For instance, we could define the “length” of an open set to be the sum of the lengths of open intervals of which it is composed. Since the collection of open sets is quite restrictive, we would like to construct a set

²That is $l(I) = b - a$ with $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, $a < b$, and $I = [a, b], (a, b), [a, b), (-\infty, b]$, etc.

function f that assigns to each set E in the collection $\mathcal{P}(\mathbb{R})$ a non-negative extended real number fE called the measure of E (i.e. $f : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_+ \cup \{\infty\}$).

Remark 1 The “ideal” properties of the set function $f : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ are: (i) fE is defined for every set $E \subset \mathbb{R}$; (ii) for an interval I , $fI = l(I)$; (iii) f is countably additive; that is, if $\{E_n\}_{n \in \mathbb{N}}$ is a collection of disjoint sets (for which f is defined), $f(\cup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} fE_n$; and (iv) f is translation invariant; that is, if E is a set for which f is defined and if $E + y$ is the set $\{x + y : x \in E\}$ obtained by replacing each point $x \in E$ by the point $x + y$, then $f(E + y) = fE$.³

Unfortunately, it is impossible to construct a set function having all four of the properties in Remark 1. As a result at least one of these four properties must be weakened.

- Following Henri Lebesgue, it is most useful to retain the last three properties (ii)-(iv) and to weaken the property in (i) so that fE need not be defined on $\mathcal{P}(\mathbb{R})$.
- It is also possible to weaken (iii) by replacing it with finite additivity (i.e., require that for each finite collection $\{E_n\}_{n=1}^N$, we have $f(\cup_{n=1}^N E_n) = \sum_{n=1}^N fE_n$).

5.1.1 Outer measure

Another possibility is to retain (i),(ii),(iv), and weaken (iii) in Remark 1 to allow countable subadditivity (i.e., $f(\cup_{n \in \mathbb{N}} E_n) \leq \sum_{n \in \mathbb{N}} fE_n$). A set function which satisfies this is called the outer measure.

Definition 327 For each set $A \subset \mathbb{R}$, let $\{I_n\}_{n \in \mathbb{N}}$ denote a countable collection of open intervals that covers A (i.e. collections such that $A \subset \cup_{n \in \mathbb{N}} I_n$) and for each such collection consider $\sum_{n \in \mathbb{N}} l(I_n)$. The **outer measure** $m^* : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is given by

$$m^*(A) = \inf_{\{I_n\}_{n \in \mathbb{N}}} \left\{ \sum_{n \in \mathbb{N}} l(I_n) : A \subset \cup_{n \in \mathbb{N}} I_n \right\}.$$

³For instance, translation invariance simply says the length of a unit interval starting at 0 should be the same as a unit interval starting at 3.

Thus, the outer measure is the least overestimate of the length of a given set A . The outer measure is well defined since each element of $\mathcal{P}(\mathbb{R})$ (i.e. subset $A \subset \mathbb{R}$) can be covered by a countable collection of open intervals which follows from Theorem 108. We establish the properties of the outer measure in the next series of theorems.

Theorem 328 (i) $m^*(A) \geq 0$. (ii) $m^*(\emptyset) = 0$. (iii) If $A \subset B$, then $m^*(A) \leq m^*(B)$ (i.e. monotonicity). (iv) $m^*(A) = 0$ for every singleton set A . (v) m^* is translation invariant.

Exercise 5.1.1 Prove Theorem 328. Theorem 2.2, p. 56 of Jain and Gupta.

The next theorem shows that we can extend the notion of length that is defined for *any subset* of \mathbb{R} .

Theorem 329 The outer measure of an interval is its length.

Proof. (Sketch) Let $\{I_n\}$ be an open covering of $[a, b]$. Then by the Heine-Borel Theorem 194 there is a finite subcollection of intervals that also covers $[a, b]$. Arrange them such that their left endpoints form an increasing sequence $a_1 < a_2 < \dots < a_n$. See Figure 5.1.1. Since $[a, b]$ is connected, intervals must overlap which means that $\cup_{i=1}^N (a_i, b_i) = (a_1, b_k)$ for some k with $1 \leq k \leq N$ and $[a, b] \subset (a_1, b_k)$. Thus $b - a \leq (b_k - a_1) \leq \sum_{n=1}^{\infty} \ell(I_n)$. ■

Definition 330 Let $\{A_n\}$ be a countable collection of sets with $A_n \subset \mathbb{R}$. We say m^* is **countably subadditive** if $m^*(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} m^* A_n$.

Theorem 331 Let $\{A_n\}$ be a countable collection of sets with $A_n \subset \mathbb{R}$. Then m^* is countably subadditive.

Proof. (Sketch) By the infimum property, for a given $\varepsilon > 0$, there is a countable collection of intervals $\{I_k^n\}_{k \in \mathbb{N}}$ covering A_n (i.e. $A_n \subset \cup_{k \in \mathbb{N}} I_k^n$) such that $\sum_{k \in \mathbb{N}} l(I_k^n) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$. Notice that $\cup_{n \in \mathbb{N}} A_n$ must be covered by $\cup_{n \in \mathbb{N}} (\cup_{k \in \mathbb{N}} I_k^n)$ which is a countable union of countable sets and hence countable. By monotonicity of m^* we have

$$m^*(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} m^*(A_n) + \varepsilon$$

since $\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$. Subadditivity follows since $\varepsilon \geq 0$ was arbitrary and we can let $\varepsilon \rightarrow 0$. ■

There are also two important corollaries that follow from Theorem 331. The first important point is that there are unbounded sets with finite outer measure.

Corollary 332 *If A is a countable set, then $m^*(A) = 0$.*

Proof. Since A is countable, it can be expressed as $\{a_1, a_2, \dots, a_n, \dots\}$. Given $\varepsilon > 0$, we can enclose each a_n in an open interval I_n with $l(I_n) = \frac{\varepsilon}{2^n}$ to get

$$m^*(A) \leq \sum_{n \in \mathbb{N}} l(I_n) = \sum_{n \in \mathbb{N}} \frac{\varepsilon}{2^n} = \varepsilon.$$

The result follows as we let $\varepsilon \rightarrow 0$. ■

One important example of this is to let $A = \mathbb{Q}$ (i.e. the rationals are a set of outer measure zero). The contrapositive of Corollary 332, that a set with outer measure different from zero is uncountable, is obviously true.

Corollary 333 *$[0, 1]$ is uncountable.*

Proof. Suppose, to the contrary, that $[0, 1]$ is countable. Then by Corollary 332, $m^*([0, 1]) = 0$ in which case $l([0, 1]) = 0$ by Theorem 329, which leads to the contradiction. ■

The converse of Corollary 332, that a set with outer measure zero is countable is not always true. To see this, consider the Cantor set F constructed in Section 3.4. In particular,

$$F = \bigcap_{n \in \mathbb{N}} F_n \left(\equiv [0, 1] \setminus \bigcup_{n \in \mathbb{N}} A_n \right)$$

where

- $A_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$
- $A_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{3}{9}, \frac{6}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$
- $A_3 = \left(\frac{1}{27}, \frac{2}{27}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{27}, \frac{8}{27}\right) \cup \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{19}{27}, \frac{20}{27}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \left(\frac{25}{27}, \frac{26}{27}\right)$,
- etc.

But

- $m^*(A_1) = \frac{1}{3}$
- $m^*(A_2) = \frac{1}{9} + \frac{1}{3} + \frac{1}{9} = 2^1 \cdot \left(\frac{1}{3}\right)^2 + 2^0 \cdot \left(\frac{1}{3}\right)^1$
- $m^*(A_3) = \frac{1}{27} + \frac{1}{9} + \frac{1}{27} + \frac{1}{3} + \frac{1}{27} + \frac{1}{9} + \frac{1}{27} = 2^2 \cdot \left(\frac{1}{3}\right)^3 + 2^1 \cdot \left(\frac{1}{3}\right)^2 + 2^0 \cdot \left(\frac{1}{3}\right)^1$

and in general

$$\begin{aligned} m^*(A_n) &= 2^{n-1} \cdot \left(\frac{1}{3}\right)^n + 2^{n-2} \cdot \left(\frac{1}{3}\right)^{n-1} + \dots + 2^1 \cdot \left(\frac{1}{3}\right)^2 + 2^0 \cdot \left(\frac{1}{3}\right)^1 \\ &= \frac{1}{3} \left[\left(\frac{2}{3}\right)^{n-1} + \left(\frac{2}{3}\right)^{n-2} + \dots + \left(\frac{1}{3}\right)^1 + 1 \right] \\ &= \frac{1}{3} \cdot \left[\frac{1 - \left(\frac{2}{3}\right)^n}{1 - \left(\frac{2}{3}\right)} \right] = 1 - \left(\frac{2}{3}\right)^n. \end{aligned}$$

Since $F_1 \supset F_2 \supset \dots \supset F_n \supset \dots$ and $m^*(F_1) = \frac{2}{3} < \infty$, by Theorem 344.

$$\begin{aligned} m^*(F) &= \lim_{n \rightarrow \infty} m^*(F_n) = \lim_{n \rightarrow \infty} m^*([0, 1] \setminus A_n) = \lim_{n \rightarrow \infty} [m^*([0, 1]) - m^*(A_n)] \\ &= 1 - \lim_{n \rightarrow \infty} m^*(A_n) = 1 - \lim_{n \rightarrow \infty} 1 - \left(\frac{2}{3}\right)^n = 1 - 1 = 0. \end{aligned}$$

Hence, the Cantor set presents an example of an uncountable set with outer measure zero.

Sets of outer measure zero provide another notion of “small” sets. From the point of view of cardinality, F is big (uncountable) while \mathbb{Q} is small (countable). From the topological point of view, F is small (nowhere dense) while \mathbb{Q} is big (dense). From the point of view of measure, both F and \mathbb{Q} are small (measure zero).

5.1.2 \mathcal{L} -measurable sets

While the outer measure has the advantage that it is defined for $\mathcal{P}(\mathbb{R})$, Theorem 331 showed that it is countably subadditive but not necessarily countably additive. In order to satisfy countable additivity, we have to restrict the domain of the function m^* to some suitable subset, call it \mathcal{L} (for Lebesgue) of $\mathcal{P}(\mathbb{R})$. The members of \mathcal{L} are called \mathcal{L} -measurable sets.

Definition 334 A set $E \subset \mathbb{R}$ is (**Lebesgue**) \mathcal{L} -measurable if $\forall A \subset \mathbb{R}$ we have $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.

The definition of \mathcal{L} -measurability says that the measurable sets are those (bounded or unbounded) which split every set (measurable or not) into two parts that are additive with respect to the outer measure.

Since $A = (A \cap E) \cup (A \cap E^c)$ and m^* is subadditive, we always have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c).$$

Thus, in order to establish that E is measurable, we need only show, for any set A , that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c). \quad (5.1)$$

Inequality (5.1) is often used in practice to determine whether a given set E is measurable where A is called the **test set**.

Since Definition 334 is symmetric in E and E^c , we have that E^c is \mathcal{L} -measurable whenever E is. Clearly, \emptyset and \mathbb{R} are \mathcal{L} -measurable.

Lemma 335 If $m^*(E) = 0$, then E is \mathcal{L} -measurable.

Proof. Let $A \subset \mathbb{R}$ be any set. Since $A \cap E \subset E$ we have $m^*(A \cap E) \leq m^*(E) = 0$. Since $A \cap E^c \subset A$ we have $m^*(A) \geq m^*(A \cap E^c) = m^*(A \cap E^c) + m^*(A \cap E)$ which follows from above. Hence E is \mathcal{L} -measurable. ■

Corollary 336 Every countable set is \mathcal{L} -measurable and its measure is zero.

Proof. From Lemma 335 and Corollary 332. ■

Exercise 5.1.2 Show that if $m^*(E) = 0$, then $m^*(E \cup A) = m^*(A)$ and that if in addition $A \subset E$, then $m^*(A) = 0$.

Lemma 337 If E_1 and E_2 are \mathcal{L} -measurable, so is $E_1 \cup E_2$.

Proof. Since E_1 and E_2 are \mathcal{L} -measurable, for any set A , we have

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c) \\ &= m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*([A \cap E_1^c] \cap E_2^c) \\ &= m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap [E_1 \cup E_2]^c) \\ &\geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^c) \end{aligned}$$

where the first equality follows from Definition 334 and the fact that E_1 is measurable, the second equality follows from the definition and taking the test set to be $A \cap E_1^c$ and E_2 is measurable, the third equality follows from simple set operations like DeMorgan's law, and the inequality follows from the subadditivity of m^* and the fact that $[A \cap E_1] \cup [A \cap E_2 \cap E_1^c] = A \cap [E_1 \cup E_2]$.⁴ But this satisfies (5.1), which is sufficient for \mathcal{L} -measurability. ■

Corollary 338 *The collection \mathcal{L} of all \mathcal{L} -measurable sets is an algebra of sets in $\mathcal{P}(\mathbb{R})$.*

Proof. Follows from Definition 81, the symmetry (w.r.t. complements) in Definition 334, and Lemma 337 ■

Lemma 339 *Let $A \subset \mathbb{R}$ be any set and $\{E_n\}_{n=1}^N$ be a finite collection of disjoint \mathcal{L} -measurable sets in \mathbb{R} . Then $m^*(A \cap [\cup_{n=1}^N E_n]) = \sum_{n=1}^N m^*(A \cap E_n)$.*

Proof. The result is clearly true for $N = 1$. Consider an induction on N . Suppose the result is true for $N - 1$. Since the E_n are disjoint, $A \cap [\cup_{n=1}^N E_n] \cap E_N = A \cap E_N$ and $A \cap [\cup_{n=1}^N E_n] \cap E_N^c = A \cap [\cup_{n=1}^{N-1} E_n]$. Then

$$\begin{aligned} m^*(A \cap [\cup_{n=1}^N E_n]) &= m^*(A \cap E_N) + m^*(A \cap [\cup_{n=1}^{N-1} E_n]) \\ &= m^*(A \cap E_N) + \sum_{n=1}^{N-1} m^*(A \cap E_n) \end{aligned}$$

where the first equality follows from Definition 334 and the second follows since the result is true for $N - 1$. ■

Corollary 340 *If $\{E_n\}_{n=1}^N$ is a finite collection of disjoint \mathcal{L} -measurable sets in \mathbb{R} , then $m^*(\cup_{n=1}^N E_n) = \sum_{n=1}^N m^* E_n$.*

Proof. Taking $A = \mathbb{R}$, the result follows from Corollary 338 and Lemma 339. ■

The result in Corollary 340 verifies that m^* restricted to \mathcal{L} is finitely additive. However, we would like to extend it to the more general case of countable additivity. First, we must show that \mathcal{L} is a σ -algebra (as discussed in section 2.6) so that $(\cup_{n=1}^{\infty} E_n) \in \mathcal{L}$ for any $\{E_n, E_n \in \mathcal{L}\}$ so that the m^* is well defined.

⁴That is, we know $m^*([A \cap E_1] \cup [A \cap E_2 \cap E_1^c]) = m^*(A \cap [E_1 \cup E_2])$ and subadditivity implies $m^*([A \cap E_1] \cup [A \cap E_2 \cap E_1^c]) \leq m^*([A \cap E_1]) + m^*([A \cap E_2 \cap E_1^c])$

Theorem 341 *The collection \mathcal{L} of all \mathcal{L} -measurable sets is a σ -algebra of sets in $\mathcal{P}(\mathbb{R})$.*

Proof. (Sketch) Let $E = \bigcup_{n \in \mathbb{N}} E_n$. First we use the fact that \mathcal{L} is an algebra: i.e. $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{L}$ and that $(\bigcup_{n=1}^N E_n)^c \supset (\bigcup_{n \in \mathbb{N}} E_n)^c = E^c$. Hence

$$m^*(A) \geq \sum_{n=1}^N m^*(A \cap E_n) + m^*(A \cap E^c).$$

By letting $N \rightarrow \infty$ and using countable subadditivity of m^* we get where the first equality follows by Definition 334, the inequality follows since $F_N^c \supset E^c$, and the last equality follows by Lemma 339. Since the left hand side of (5.13) is independent of N , letting $N \rightarrow \infty$ we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

■

Definition 342 *The set function $m : \mathcal{L} \rightarrow \mathbb{R}_+ \cup \{\infty\}$, obtained by restricting the functions m^* to the σ -algebra $\mathcal{L} \subset \mathcal{P}(\mathbb{R})$ is called the **Lebesgue measure**. That is, $m = m^*|_{\mathcal{L}}$.⁶*

The next result shows that after relaxing point (i) in Remark 1 we can satisfy property (iii) with the Lebesgue measure.

Theorem 343 *If $\{E_n\}_{n \in \mathbb{N}}$ is a countable collection of disjoint sets in \mathbb{R} , then $m(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} m(E_n)$.*

Proof. Since $(\bigcup_{n=1}^N E_n) \subset (\bigcup_{n \in \mathbb{N}} E_n)$, $\forall N \in \mathbb{N}$, and both sets are \mathcal{L} -measurable by Theorems 338 and 341, we have $m(\bigcup_{n \in \mathbb{N}} E_n) \geq m(\bigcup_{n=1}^N E_n) = \sum_{n=1}^N m(E_n)$ where the equality follows by Corollary 340. Since the left hand of the inequality is independent of N , letting $N \rightarrow \infty$ we have $m(\bigcup_{n \in \mathbb{N}} E_n) \geq \sum_{n \in \mathbb{N}} m(E_n)$. Since the reverse inequality holds by countable subadditivity in Theorem 331, the result follows. ■

The next property will be useful in proving certain convergence properties in upcoming sections and can be viewed as a continuity property of the Lebesgue measure.

⁵Recall by DeMorgan's Law that $F_N^c = [\bigcup_{n=1}^N E_n]^c = \bigcap_{n=1}^N E_n^c$.

⁶This follows from Definition 56.

Theorem 344 *Let $\langle E_n \rangle$ be an infinite decreasing sequence of \mathcal{L} -measurable sets (i.e. $E_{n+1} \subset E_n, \forall n$). Let mE_1 be finite. Then $m(\cap_{i=1}^{\infty} E_i) = \lim_{n \rightarrow \infty} m(E_n)$.*

Proof. Since \mathcal{L} is a σ -algebra, $\cap_{n=1}^{\infty} E_n \in \mathcal{L}$. The set $E_1 \setminus \cap_{n=1}^{\infty} E_n$ can be written as the union of mutually disjoint sets $\{E_n \setminus E_{n+1}\}$ (see Figure 5.1.2)

$$E_1 \setminus \cap_{n=1}^{\infty} E_n = (E_1 \setminus E_2) \cup (E_2 \setminus E_3) \cup \dots \cup (E_n \setminus E_{n+1}) \cup \dots$$

Then using countable additivity of m we have

$$\begin{aligned} m(E_1) - m(\cap_{n=1}^{\infty} E_n) &= m(E_1 \setminus \cap_{n=1}^{\infty} E_n) = m(\cup_{n=1}^{\infty} E_n \setminus E_{n+1}) \\ &= \sum_{n=1}^{\infty} m(E_n \setminus E_{n+1}) = \sum_{n=1}^{\infty} [m(E_n) - m(E_{n+1})] \\ &= m(E_1) - \lim_{n \rightarrow \infty} m(E_n). \end{aligned}$$

Comparing the beginning and end we have $m(\cap_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} m(E_n)$. ■

5.1.3 Lebesgue meets borel

Now that we know that \mathcal{L} is a σ -algebra, we might ask what type of sets belong in \mathcal{L} ? For example, are open and/or closed sets in \mathcal{L} ?

Lemma 345 *The interval (a, ∞) is \mathcal{L} -measurable.*

Proof. (Sketch) Take any set A . The open ray (a, ∞) splits A into two disjoint parts $A_1 = (a, \infty) \cap A$ and $A_2 = (-\infty, a] \cap A$. According to (5.1), it suffices to show $m^*(A) \geq m^*(A_1) + m^*(A_2)$. For $\varepsilon > 0$, there is a countable collection $\{I_n\}$ of open intervals which covers A satisfying $\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$ by the infimum property in Definition 327. Again, (a, ∞) splits each interval $I_n \in \{I_n\}$ into two disjoint intervals I'_n and I''_n . Clearly $\{I'_n\}$ covers A_1 , $\{I''_n\}$ covers A_2 , and $\sum_n \ell(I_n) = \sum_n \ell(I'_n) + \sum_n \ell(I''_n)$. By monotonicity and subadditivity of m^* we have

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq m^*(\cup_n I'_n) + m^*(\cup_n I''_n) \\ &\leq \sum_{n=1}^{\infty} \ell(I'_n) + \sum_{n=1}^{\infty} \ell(I''_n) \\ &= \sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon. \end{aligned}$$

But since $\varepsilon > 0$ was arbitrary, the result follows. ■

The next result shows that every open or closed set in \mathbb{R} is \mathcal{L} -measurable.

Theorem 346 *Every Borel set is \mathcal{L} -measurable.*

Proof. The result follows from Theorem 124 that the collection of all open rays generates \mathcal{B} ■

Thus, the Lebesgue measure m is defined for Borel sets. Hence we can work with sets we know a lot about. While it is beyond the scope of the book, we note that there are examples of sets that show $\mathcal{B} \subsetneq \mathcal{L}$ and $\mathcal{L} \subsetneq \mathcal{P}(\mathbb{R})$.⁷

The next theorem gives a useful characterization of measurable sets. It asserts that a measurable set can be “approximated” by open and closed sets. See Figure 5.1.3.

Theorem 347 *Let E be a measurable subset of \mathbb{R} . Then for each $\varepsilon > 0$, there exists an open set G and a closed set F such that $F \subset E \subset G$ and $m(G \setminus F) < \varepsilon$.*

Proof. Since E is measurable, $m(E) = m^*(E)$. We use the infimum property for sets E and E^c . Given $\frac{\varepsilon}{2}$, there exist open sets G and H (remember that the union of open intervals is an open set) such that

$$\begin{aligned} E &\subset G \text{ and } m(G) < m(E) + \frac{\varepsilon}{2} \\ \text{and } E^c &\subset H \text{ and } m(H) < m(E^c) + \frac{\varepsilon}{2}. \end{aligned}$$

Set $F = H^c$. See Figure 5.1.4. Then by the properties of complements, we have that F is closed, $E \supset F$, and $m(E) - m(F) < \frac{\varepsilon}{2}$. Thus we have $F \subset E \subset G$ and

$$m(G \setminus F) = m(G) - m(F) = m(G) - m(E) + m(E) - m(F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The first equality is due to additivity and the second is simply an identity. ■

5.1.4 \mathcal{L} -measurable mappings

Before we actually begin to integrate a mapping, we must know that a given mapping is integrable. We break this topic up into two parts: functions and correspondences.

⁷For an example of a non-measurable set see p. 289 of Carothers (2000).

functions

Roughly speaking, a function is integrable if its behavior is not too irregular and if the values it takes on are not too large too often. We now introduce the notion of measurability which gives precisely the conditions required for integrability, provided the function is not too large.

Definition 348 Let $f : E \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ where E is \mathcal{L} -measurable. Then f is \mathcal{L} -measurable if the set $\{x \in E : f(x) \leq \alpha\} \in \mathcal{L}, \forall \alpha \in \mathbb{R}$.

It is clear from the above definition that there is a close relation between measurability of a function and the measurability of the inverse image set. In particular, it can be shown that f is \mathcal{L} -measurable iff for any closed set $G \subset \mathbb{R}$, inverse image $f^{-1}(G)$ is a measurable set. See Figure 5.1.4.1.

As α varies, the behavior of the set $\{x \in E : f(x) \leq \alpha\}$ describes how the values of the function f are distributed. The smoother is f , the smaller the variety of inverse images which satisfy the restriction on f .

Example 349 Consider an *indicator or characteristic function* $\chi_A : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} .$$

with $A \subset \mathbb{R}$. Then χ_A is \mathcal{L} -measurable iff $A \in \mathcal{L}$. To see this, note that

$$\{x \in \mathbb{R} : \chi_A(x) \leq \alpha\} = \begin{cases} \emptyset & \text{if } \alpha < 0 \\ \mathbb{R} \setminus A & \text{if } 0 \leq \alpha < 1 \\ \mathbb{R} & 1 \leq \alpha \end{cases} .$$

But $\{\emptyset, A^c, \mathbb{R}\} \in \mathcal{L}$. Figure 5.1.4.2a.

Example 350 Let the function $f : [0, 1] \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } 0 < x < 1 \\ 2 & \text{if } x = 1 \end{cases} .$$

Notice that this function is neither continuous nor monotone. To see that f is \mathcal{L} -measurable, note that

$$\{x \in \mathbb{R} : f(x) \leq \alpha\} = \begin{cases} \emptyset & \alpha < 1 \\ \{0\} & \alpha = 1 \\ [\frac{1}{\alpha}, 1] \cup \{0\} & 1 < \alpha < 2 \\ [\frac{1}{\alpha}, 1] \cup \{0\} & 2 \leq \alpha \end{cases} .$$

Again, all these sets are in \mathcal{L} . See Figure 5.1.4.2b. This example shows that an \mathcal{L} -measurable function need not be continuous.

The next result establishes that there are many criteria by which to establish measurability of a function.

Theorem 351 *Let $f : E \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ where E is \mathcal{L} -measurable. Then the following statements are equivalent: (i) $\{x \in E : f(x) \leq \alpha\}$ is \mathcal{L} -measurable $\forall \alpha \in \mathbb{R}$; (ii) $\{x \in E : f(x) > \alpha\}$ is \mathcal{L} -measurable $\forall \alpha \in \mathbb{R}$; (iii) $\{x \in E : f(x) \geq \alpha\}$ is \mathcal{L} -measurable $\forall \alpha \in \mathbb{R}$; (iv) $\{x \in E : f(x) < \alpha\}$ is \mathcal{L} -measurable $\forall \alpha \in \mathbb{R}$; These statements imply (v) $\{x \in E : f(x) = \alpha\}$ is \mathcal{L} -measurable $\forall \alpha \in \mathbb{R} \cup \{-\infty, \infty\}$.*

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i) can be established from

$$\begin{aligned} \{x \in E : f(x) > \alpha\} &= E \setminus \{x \in E : f(x) \leq \alpha\} \\ \{x \in E : f(x) \geq \alpha\} &= \bigcap_{n=1}^{\infty} \left\{ x \in E : f(x) > \alpha - \frac{1}{n} \right\} \\ \{x \in E : f(x) < \alpha\} &= E \setminus \{x \in E : f(x) \geq \alpha\} \\ \{x \in E : f(x) \leq \alpha\} &= \bigcap_{n=1}^{\infty} \left\{ x \in E : f(x) < \alpha + \frac{1}{n} \right\} \end{aligned}$$

where each operation follows since \mathcal{L} is a σ -algebra (which is closed under complementation and countable intersection).

Next, if $\alpha \in \mathbb{R}$, then $\{x \in E : f(x) = \alpha\} = \{x \in E : f(x) \leq \alpha\} \cap \{x \in E : f(x) \geq \alpha\}$. If $\alpha = \infty$, then since $\{x \in E : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) \geq n\}$ we have (iii) \Rightarrow (v). A similar result holds for $\alpha = -\infty$. where the first line follows since \mathcal{L} is a σ -algebra (which is closed under countable intersection) and the second follows since the difference of two measurable sets is measurable. ■

Next we present some properties of \mathcal{L} -measurable functions

Lemma 352 (i) *If f is an \mathcal{L} -measurable function on the set E and $E_1 \subset E$ is an \mathcal{L} -measurable set, then f is an \mathcal{L} -measurable function on E_1 . (ii) If f and g are \mathcal{L} -measurable functions on E , then the set $\{x \in E : f(x) < g(x)\}$ is \mathcal{L} -measurable.*

Proof. (i) follows since $\{x \in E_1 : f(x) > \alpha\} = \{x \in E : f(x) > \alpha\} \cap E_1$ and the intersection of two \mathcal{L} -measurable sets is measurable. (ii) Define

$A_q = \{x \in E : f(x) < q < g(x)\}$ with $q \in \mathbb{Q}$ whose existence is guaranteed by Theorem 100. Then $A_q = \{x \in E : f(x) < q\} \cap \{x \in E : g(x) > q\}$ and $\{x \in E : f(x) < g(x)\} = \cup_{q \in \mathbb{Q}} A_q$, which is a countable union of \mathcal{L} -measurable sets. ■

The next theorem establishes that certain operations performed on \mathcal{L} -measurable functions preserve measurability.

Theorem 353 *Let f and g be \mathcal{L} -measurable functions on E and c be a constant. Then the following functions are \mathcal{L} -measurable: (i) $f \pm c$; (ii) cf ; (iii) $f \pm g$; (iv) $|f|$; (v) f^2 ; (vi) fg .*

Proof. (i) $\{x \in E : f(x) \pm c > \alpha\} = \{x \in E : f(x) > \alpha'\}$ with $\alpha' = \alpha \mp c$ so $f \pm c$ is \mathcal{L} -measurable when f is.

(ii) If $c = 0$, then cf is \mathcal{L} -measurable since any constant function is \mathcal{L} -measurable. Otherwise

$$\{x \in E : cf(x) > \alpha\} = \begin{cases} \{x \in E : f(x) > \alpha'\} & \text{if } c > 0 \\ \{x \in E : f(x) < \alpha'\} & \text{if } c < 0 \end{cases}$$

with $\alpha' = \frac{\alpha}{c}$ is \mathcal{L} -measurable since f is \mathcal{L} -measurable.

(iii) $\{x \in E : f(x) + g(x) > \alpha\} = \{x \in E : f(x) > \alpha - g(x)\}$. Since $\alpha - g$ is \mathcal{L} -measurable by (i) and (ii), then $f + g$ is \mathcal{L} -measurable by Lemma 352.

(iv) Follows since

$$\{x \in E : |f(x)| > \alpha\} = \begin{cases} E & \text{if } \alpha < 0 \\ \{x \in E : f(x) > \alpha\} \cup \{x \in E : f(x) < -\alpha\} & \text{if } \alpha \geq 0 \end{cases}$$

and both sets on the rhs are \mathcal{L} -measurable since f is \mathcal{L} -measurable.

(v) Follows from

$$\{x \in E : (f(x))^2 > \alpha\} = \begin{cases} E & \text{if } \alpha < 0 \\ \{x \in E : |f(x)| > \alpha\} & \text{if } \alpha \geq 0 \end{cases}$$

and (iv).

(vi) Follows from the identity $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$ and (ii), (iii), (v). ■

Parts (ii) and (iii) of Theorem 353 imply that scaled linear combinations of indicator functions defined on measurable sets are themselves measurable functions. This type of function, known as a simple function, will play an important role in approximating a given function. As we will see in the next

section, unlike the standard (Riemann) way of approximating the integral of f by calculating the area under f based on partitions of the domain, in this chapter we will be approximating the integral of f by calculating the area under f based on partitions of the range. See Figure 5.1.4.2c.

Definition 354 A function $\varphi : E \rightarrow \mathbb{R}$ given by

$$\varphi(x) = \sum_{i=1}^n a_i \chi_{E_i}(x) \quad (5.2)$$

is called a **simple function** if there is a finite collection $\{E_1, \dots, E_n\}$ of disjoint \mathcal{L} -measurable sets with $\cup_{i=1}^n E_i = E$ and a finite set of real numbers $\{a_1, \dots, a_n\}$ such that $a_i = \varphi(x), \forall x \in E_i$ for $i = 1, \dots, n$ where $\chi_{E_i}(x)$ is an indicator function introduced in Example 349. The right hand side of (5.2) is called the representation of φ .

We note that the real numbers $\{a_i\}$ and the sets $\{E_i\}$ in this representation are not uniquely determined as the next example shows.

Example 355 Let $\{E_1, E_2, E_3\}$ be disjoint subsets of an \mathcal{L} -measurable set E . Consider the two simple functions $2\chi_{E_1} + 5\chi_{E_2} + 2\chi_{E_3}$ and $2\chi_{E_1 \cup E_3} + 5\chi_{E_2}$. Clearly these two simple functions are equal. Notice that the coefficients in the first representation are not distinct. Since a simple function obtains only a finite number of values $\{a_1, \dots, a_n\}$ on E we can construct the inverse image sets $\{A_i\}$ as $A_i = \{x \in E : x = \varphi^{-1}(\{a_i\})\}$, $i = 1, \dots, n$, $n \in \mathbb{R}$. In this example, $A_1 = \{x \in E : x = \varphi^{-1}(\{2\})\} = E_1 \cup E_3$ and $A_2 = \{x \in E : x = \varphi^{-1}(\{5\})\} = E_2$, both of which are \mathcal{L} -measurable, disjoint sets. To avoid such problems with non-uniqueness, we use this construction to define the **canonical (or standard) representation** of $\varphi : E \rightarrow \mathbb{R}$ by $\varphi(x) = \sum_{i=1}^k a_i \chi_{A_i}(x)$ where the finite collection $\{A_1, \dots, A_k\}$ of \mathcal{L} -measurable sets are disjoint with $\cup_{i=1}^k A_i = E$ and the finite set of real numbers $\{a_1, \dots, a_k\}$ are distinct and nonzero.

The next theorem and exercise provide sufficient conditions for \mathcal{L} -measurability.

Theorem 356 A continuous function defined on an \mathcal{L} -measurable set is \mathcal{L} -measurable.

Proof. Let f be a continuous function defined on E (which is \mathcal{L} -measurable). Consider the set $A = \{x \in E : f(x) > \alpha\}$ which is the inverse image of the open ray (α, ∞) . Since f is continuous, Theorem 248 implies $f^{-1}((\alpha, \infty))$ is open and hence \mathcal{L} -measurable. ■

Exercise 5.1.3 Show that any monotone function $f : \mathbb{R} \rightarrow \mathbb{R}$ is \mathcal{L} -measurable.

Next we consider how sequences of \mathcal{L} -measurable functions behave.

Theorem 357 Let $\langle f_n \rangle$ be a sequence of functions on a common domain E . Then the functions $\max\{f_1, \dots, f_n\}$, $\min\{f_1, \dots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, and $\liminf_n f_n$ are all \mathcal{L} -measurable.

Proof. If $g(x) = \max\{f_1(x), \dots, f_n(x)\}$, then $\{x \in E : g(x) > \alpha\} = \cup_{i=1}^n \{x \in E : f_i(x) > \alpha\}$ and \mathcal{L} -measurability of each f_i implies g is \mathcal{L} -measurable. Similarly, if $h(x) = \sup_n f_n(x)$, then $\{x \in E : h(x) > \alpha\} = \cup_{i=1}^{\infty} \{x \in E : f_i(x) > \alpha\}$. A similar argument, along with the fact that $\inf_n f_n = -\sup_n(-f_n)$ and (ii) of Theorem 353, establishes the corresponding statements for \inf . To establish the last results, note that $\liminf_n f_n = -\limsup_n(-f_n) = \sup_n(\inf_{k \geq n} f_k)$. ■

Using the above results, for any function $f : E \rightarrow \mathbb{R}$, we can construct non-negative functions $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. The function f is \mathcal{L} -measurable iff both f^+ and f^- are \mathcal{L} -measurable. It is also easy to verify that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. See Figure 5.1.4.3.

Corollary 358 (i) If $\langle f_n \rangle$ is a sequence of \mathcal{L} -measurable functions converging pointwise to f on E , then f is \mathcal{L} -measurable. (ii) The set of points on which $\langle f_n \rangle$ converges is \mathcal{L} -measurable.

Proof. (i) Since $f_n \rightarrow f$, we have $\limsup_n f_n = \liminf_n f_n = f$ by Theorem 321 and the result thus follows from Theorem 357 above. (ii) From (i), the set $\{x \in E : \limsup_n f_n - \liminf_n f_n = 0\}$ is \mathcal{L} -measurable by (v) of Theorem 351. ■

In some cases, two functions may be “almost” the same in the sense of \mathcal{L} -measurability. The next definition helps us make that precise.

Definition 359 A property is said to hold **almost everywhere (a.e.)** if the set of points where it fails to hold is a set of measure zero.

Example 360 Let $f : [0, 1] \rightarrow \{0, 1\}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

known as the Dirichlet function. While this function is famous since it is everywhere discontinuous (which we will use in Section 5.2.1, here we simply use it to illustrate the concept of almost everywhere. In particular, $f(x) = 0$ a.e. since $\{x \in [0, 1] : f(x) \neq 0\} = \{x \in [0, 1] : x \in \mathbb{Q}\}$ and $m(\{x \in [0, 1] : x \in \mathbb{Q}\}) = 0$ which follows from the countability of the rationals established in Example 77 and Corollary 336..

Theorem 361 Let f and g have domain E and let f be an \mathcal{L} -measurable function. If $f = g$ a.e., then g is measurable.

Proof. Let $D = \{x \in E : f(x) \neq g(x)\}$. Then $mD = 0$ by assumption. Let $\alpha \in \mathbb{R}$, and consider

$$\begin{aligned} \{x \in E : g(x) > \alpha\} &= \{x \in E \setminus D : f(x) > \alpha\} \cup \{x \in D : g(x) > \alpha\} \\ &= [\{x \in E : f(x) > \alpha\} \setminus \{x \in D : g(x) \leq \alpha\}] \cup \{x \in D : g(x) > \alpha\} \end{aligned}$$

Since f is \mathcal{L} -measurable, the first set is \mathcal{L} -measurable. Furthermore, since the other two sets are contained in D , which has measure 0, they are \mathcal{L} -measurable by Lemma 335 and Exercise 5.1.2. ■

Now we consider a weaker notion of continuity than considered in Theorem 356.

Theorem 362 If a function f defined on E (which is \mathcal{L} -measurable) is continuous a.e., then f is \mathcal{L} -measurable on E .

Proof. Follows from Theorem 356. ■

Theorem 362 thus establishes the sufficient condition such that the discontinuous and non-monotone function in Example 350 is \mathcal{L} -measurable.

Now we consider a weaker version of convergence than considered in Corollary 358.

Definition 363 A sequence $\langle f_n \rangle$ of functions defined on E is said to **converge a.e.** to a function f if $\lim_{n \rightarrow \infty} f_n(x) = f(x), \forall x \in E \setminus E_1$ where $E_1 \subset E$ with $mE_1 = 0$.

Theorem 364 *If a sequence $\langle f_n \rangle$ of \mathcal{L} -measurable functions converges a.e. to the function f , then f is \mathcal{L} -measurable.*

Proof. Follows from Corollary 358. ■

Example 365 *Let $\langle f_n \rangle$ be given by $\langle x^n \rangle$ on $[0, 1]$ which converges pointwise to*

$$f = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

and f is \mathcal{L} -measurable since it is the constant (zero) function almost everywhere.

The next theorem establishes that if a sequence of functions converges pointwise, then we can isolate a set of points of arbitrarily small measure such that on the complement of that set the convergence is uniform.

Theorem 366 *Let E be an \mathcal{L} -measurable set with $mE < \infty$ and $\langle f_n \rangle$ be a sequence of \mathcal{L} -measurable functions defined on E . Let $f : E \rightarrow \mathbb{R}$ be such that $\forall x \in E, f_n(x) \rightarrow f(x)$. Then given $\varepsilon > 0$ and $\delta > 0, \exists$ an \mathcal{L} -measurable set $A \subset E$ with $m(A) < \delta$ and $\exists N$ such that for $n \geq N$ and $x \notin A$ we have $|f_n(x) - f(x)| < \varepsilon$.*

Proof. Let $G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$ and $E_k = \cup_{n=k}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq k\}$. Thus $E_{k+1} \subset E_k$ and for each $x \in E$ there must be some set E_k such that $x \notin E_k$ otherwise we would violate the assumption $f_n(x) \rightarrow f(x), \forall x \in E$. Thus $\langle E_k \rangle$ is a decreasing sequence of \mathcal{L} -measurable sets for which $\cap_{k=1}^{\infty} E_k = \emptyset$ so that by Theorem 344 we have $\lim_{k \rightarrow \infty} mE_k = 0$. Hence given $\delta > 0, \exists N$ such that $mE_N < \delta$; that is, $m\{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\} < \delta$. If we write A for this E_N , then $mA < \delta$ and

$$E \setminus A = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon, \forall n \geq N\}.$$

■

The next theorem says that for any \mathcal{L} -measurable function f there exists a sequence of “nice” functions (more specifically simple functions) that converge pointwise to f . Moreover, on the subdomain where f is bounded, this convergence is uniform. This means that a bounded measurable function can be approximated by a simple function.

Theorem 367 *Let f be an \mathcal{L} -measurable function defined on a set E . Then there exists a sequence $\langle f_n \rangle$ of simple functions which converges pointwise to f on E and converges uniformly to f on any set where f is bounded. Furthermore, if $f \geq 0$, then $\langle f_n \rangle$ can be chosen such that $0 \leq f_n \leq f_{n+1}$, $\forall n \in \mathbb{N}$.*

Proof. (Sketch) We can assume that $f \geq 0$. If not, then let $f = f^+ - f^-$ where f^+ and f^- are non-negative. For $n \in \mathbb{N}$, we divide the range of f (which can be unbounded) into two parts: $[0, 2^n)$ and $[2^n, \infty)$. See Figure 5.1.4.4. Then divide $[0, 2^n)$ into $2^{2^n} - 1$ equal parts. Let F_n be the inverse image of $[2^n, \infty)$ and $E_{n,k}$ be the inverse images of $[k2^{-n}, (k+1)2^{-n}]$ for $k = 0, 1, \dots, 2^{2^n} - 1$. Since f is measurable, F_n and $E_{n,k}$ are measurable. Define a simple function

$$\varphi_n = 2^n \chi_{F_n} + \sum_{k=0}^{2^{2^n}-1} k2^{-n} \chi_{E_{n,k}}. \quad (5.3)$$

Note that $0 \leq \varphi_n \leq f$ and $0 \leq f - \varphi_n \leq 2^{-n}$ on $\cup_{k=1}^{2^{2^n}-1} E_{n,k}$. For any $x \in E$, there exists n large enough such that $f(x) < 2^n$. Hence $x \in \cup E_{n,k}$ implies that $f(x) - \varphi_n(x) \leq 2^{-n}$ and thus $\varphi_n(x) \rightarrow f(x)$. Moreover, if f is bounded, there exists an n large enough such that $E = \cup_{k=1}^{2^{2^n}-1} E_{n,k}$ and $f(x) - \varphi_n(x) < \frac{1}{n}$ for each $x \in E$, thus $\langle \varphi_n \rangle$ converges uniformly to f . ■

Exercise 5.1.4 *Show that φ_n increases in (5.3). Hint: $E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+n}$.*

The next definition will be useful in Chapter FS.

Definition 368 *Let f be an \mathcal{L} -measurable function. Then $\inf\{\alpha \in \mathbb{R} : f \leq \alpha \text{ a.e.}\}$ is called the **essential supremum** of f , denoted $\text{ess sup } f$, and $\sup\{\alpha \in \mathbb{R} : f \geq \alpha \text{ a.e.}\}$ is called the **essential infimum** of f , denoted $\text{ess inf } f$.*

Example 369 *Let $f : [0, 1] \rightarrow \{-1, 0, 1\}$ be given by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}_{++} \\ 0 & \text{if } x \text{ is irrational} \\ -1 & \text{if } x \in \mathbb{Q}_- \end{cases}$$

which is a simple generalization of the Dirichlet function. Given the results in Example 360 we have $\text{ess sup } f = \text{ess inf } f = 0$.

correspondences

Let $\Gamma : X \rightrightarrows Y$ be a correspondence where $X = \mathbb{R}$ or a subset of \mathbb{R} equipped with the Lebesgue measure and $\mathcal{L}(X)$ is a σ -algebra of all \mathcal{L} -measurable subsets of X and Y is a complete, separable metric space. We will introduce the concept of measurability of correspondences the same way we defined measurability of single-valued functions (i.e. through inverse images). We know a function $f : X \rightarrow Y$ is \mathcal{L} -measurable if $f^{-1}(V)$ is \mathcal{L} -measurable for every open set $V \subset Y$ or equivalently $f^{-1}(U)$ is \mathcal{L} -measurable for every closed set $U \subset Y$.

Definition 370 Consider a measurable space (X, \mathcal{L}) where $X \subset \mathbb{R}$ (or $X = \mathbb{R}$), Y is a complete separable metric space Y , and $\Gamma : X \rightrightarrows Y$ is a closed-value correspondence. Γ is **measurable** if the inverse image of each open set is a \mathcal{L} -measurable set. That is, for every open subset $V \subset Y$ we have $\Gamma^{-1}(V) = \{x \in X : \Gamma(x) \cap V \neq \emptyset\} \in \mathcal{L}$.

Notice that measurability is defined only for closed valued correspondences.

Given a correspondence $\Gamma : X \rightrightarrows Y$ we can ask under what conditions there exists a measurable selection of Γ (i.e. a single-valued, \mathcal{L} -measurable function $f : X \rightarrow Y$ such that $f(x) \in \Gamma(x)$ for all $x \in X$). The following theorem says that every \mathcal{L} -measurable correspondence has a measurable selection provided the spaces X and Y have certain properties.

Theorem 371 (Measurable Selection) Let (X, \mathcal{L}) be a Lebesgue measurable space, let Y be a complete separable metric space, and let $\Gamma : X \rightrightarrows Y$ be a \mathcal{L} -measurable, closed valued correspondence. Then there exists a measurable selection of Γ .

Proof. (Sketch) By induction, we will define a sequence of measurable functions $f_n : X \rightarrow Y$ such that

- (i) $f_n(z)$ is sufficiently close to $\Gamma(z)$ (i.e. $d(f_n(z), \Gamma(z)) < \frac{1}{2^n}$) and
- (ii) $f_n(z)$ and $f_{n+1}(z)$ are sufficiently close to each other (i.e. $d(f_{n+1}(z), f_n(z)) \leq \frac{1}{2^{n-1}}$ on X for all n).

Then we are done, since from (ii) it follows that $\langle f_n(z) \rangle$ is Cauchy for each z and due to completeness of Y there exists a function $f : X \rightarrow Y$ such that $f_n(z) \rightarrow f(z)$ on X pointwise and by Corollary 358 the pointwise limit f of a sequence of measurable functions is measurable. Hence we take f

as a measurable selection. Condition (i) guarantees that $f(z) \in \Gamma(z), \forall z \in X$ (here we use the fact that $\Gamma(z)$ is closed and $d(f(z), \Gamma(z)) = 0$ implies $f(z) \in \Gamma(z)$) by Exercise 4.1.3

Now we construct a sequence $\langle f_n \rangle$ of measurable functions satisfying (i) and (ii). Let $\{y_n, n \in \mathbb{N}\}$ be a dense set in Y (since Y is separable such a countable set exists). Define $f_k(z) = y_p$ where p is the smallest integer such that the ball with center at y_p with radius $\frac{1}{k}$ has non-empty intersection with $\Gamma(z)$. See Figure 5.1.4.5. It can be shown that f_k is measurable and $\langle f_k \rangle$ satisfy (i) and (ii). ■

How is measurability of a correspondence related to upper or lower hemicontinuity? We would expect that hemicontinuity implies measurability and we now show that this is true (a result similar to that for functions in Theorem 356. In the case of lower hemicontinuity we get the result immediately.

Lemma 372 *Under the assumptions of Theorem 371 if $\Gamma : X \rightarrow Y$ is lhc, then Γ is measurable.*

Proof. Since Γ is lhc, then $f^{-1}(V)$ is open for $V \subset Y$ open. Since open sets are \mathcal{L} -measurable, then $f^{-1}(V) \in \mathcal{L}$ so that f is \mathcal{L} -measurable. ■

To show that uhc implies measurability, we show that open sets can be replaced by closed sets in Definition 370.

Lemma 373 *Under the assumption of the Theorem 371, $\Gamma : X \rightarrow Y$ is measurable iff $f^{-1}(U)$ is \mathcal{L} -measurable for every closed subset $U \subset Y$.*

Proof. " \Leftarrow " Let V be an open subset of Y . Define the closed sets $C_n = \{x \in Y, d(x, Y \setminus V) \geq \frac{1}{n}\}$. Then $V = \cup_{n \in \mathbb{N}} C_n$. Consequently $\Gamma(x) \cap V \neq \emptyset$ iff $\Gamma(x) \cap C_n \neq \emptyset$ for some n . This yields $\Gamma^{-1}(V) = \cup_{n \in \mathbb{N}} \Gamma^{-1}(C_n) \in \mathcal{L}$ because $\Gamma^{-1}(C_n) \in \mathcal{L}$ by assumption (because C_n is closed) and $\cup_{n \in \mathbb{N}} \Gamma^{-1}(C_n) \in \mathcal{L}$ (because \mathcal{L} is σ -algebra).

" \Rightarrow " We omit this direction since it would require introducing measurability on the Cartesian product $X \times Y$. [See Aubin-Frankowska Section 8.3 pg.319]. ■

Lemma 374 *Under the assumption of Theorem 371, if $\Gamma : X \rightarrow Y$ is uhc then Γ is measurable.*

Proof. Since Γ is uhc, then $f^{-1}(U)$ is closed for U closed and closed sets are \mathcal{L} -measurable. Hence $f^{-1}(U) \in \mathcal{L}$. Thus f is \mathcal{L} -measurable by Lemma 373. ■

5.2 Lebesgue Integration

In introductory calculus classes, you were introduced to Riemann integration. While simple, it has many defects. First, the Riemann integral of a function is defined on a closed interval and cannot be defined on an arbitrary set. Second, a function is Riemann integrable if it is continuous or continuous almost everywhere. The set of continuous functions, however is relatively small. Third, given a sequence of Riemann integrable functions converging to some function, the limit of the sequence of the integrated function may not be the Riemann integral of the limit function. In fact the Riemann integral of the limit function may not even exist. These defects are absent in Lebesgue integration. To see these problems we begin by briefly reviewing the Riemann integral.

5.2.1 Riemann integrals

Consider a bounded function $f : [a, b] \rightarrow \mathbb{R}$ and a partition $P = \{a = x_0 < x_1 < \dots < x_{n-1} < x_n = b\}$ of $[a, b]$. Let Υ be the set of all possible partitions. For each P , define the sums

$$S(P) = \sum_{i=1}^n (x_i - x_{i-1})H_i \text{ and } s(P) = \sum_{i=1}^n (x_i - x_{i-1})h_i$$

where $H_i = \sup\{f(x) : x \in (x_{i-1}, x_i]\}$ and $h_i = \inf\{f(x) : x \in (x_{i-1}, x_i]\}$,

$\forall i = 1, \dots, n$. The sums $S(P)$ and $s(P)$ are known as step functions. See Figure 5.2.1.1. Then the **upper Riemann integral** of f over $[a, b]$ is defined by $R^u \int_a^b f(x)dx = \inf_{P \in \Upsilon} S(P)$ and the **lower Riemann integral** of f over $[a, b]$ is defined by $R^l \int_a^b f(x)dx = \sup_{P \in \Upsilon} s(P)$. If $R^u \int_a^b f(x)dx = R^l \int_a^b f(x)dx$, then we say the Riemann integral exists and denote it $R \int_a^b f(x)dx$.

We state without proof (since it would take us far afield) the following Proposition which characterizes the “class” of Riemann integrable functions.⁸

Proposition 375 *A bounded function is Riemann integrable iff it is continuous almost everywhere.*

We next provide explicit examples of functions that are and are not Riemann integrable.

⁸See Jain and Gupta (1986), Appendix 1.

Example 376 Consider the Riemann integral of Dirichlet's function introduced in Example 360. Then $R^u \int_0^1 f(x)dx = 1$ and $R^l \int_0^1 f(x)dx = 0$ so the Riemann integral does not exist. Intuitively, this is because in any partition P , however fine, there are both rational and irrational numbers which follows from the density of both sets established in Example 154. Formally, to see that the Dirichlet function (while bounded) is not continuous anywhere (and hence does not satisfy the requirements of the Proposition 375), consider the following argument. If $q \in \mathbb{Q} \cap [0, 1]$, let $\langle x_n \rangle$ be a sequence of irrational numbers converging to q (the existence of such a sequence follows from Theorem 102). Since $f(x_n) = 0, \forall n \in \mathbb{N}$, the sequence $\langle f(x_n) \rangle$ does not converge to $f(q) = 1$ so f is not continuous at $a \in \mathbb{Q}$. Similarly, if ι is an irrational number, let $\langle y_n \rangle$ be a sequence of rational numbers converging to ι (the existence of such a sequence again follows from Theorem 102). Since $f(y_n) = 1, \forall n \in \mathbb{N}$, the sequence $\langle f(y_n) \rangle$ does not converge to $f(\iota) = 0$ so f is not continuous at $\iota \in \mathbb{R} \setminus \mathbb{Q}$. See Figure 5.2.1.2.

Example 377 Next consider the Riemann integral of $f : [0, 1] \rightarrow \{0, 1\}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} .$$

Hence this function takes on the value 1 on the rationals $\{\frac{1}{n}, n \in \mathbb{N}\}$ rather than the bigger set $\mathbb{Q} = \{\frac{m}{n}, m, n \in \mathbb{N}\}$. We begin by noting that one can show that $\{\frac{1}{n}, n \in \mathbb{N}\}$ is not dense in $[0, 1]$. As in the preceding example, it is simple to show that f is discontinuous at $\{\frac{1}{n}, n \in \mathbb{N}\}$. On the other hand, f is continuous at $D = [0, 1] \setminus \{\frac{1}{n}, n \in \mathbb{N}\}$. To see this, let $x \in D \setminus \{0\}$. Then $\exists n \in \mathbb{N}$ such that $x \in (\frac{1}{n+1}, \frac{1}{n})$. Let $\delta = \frac{1}{2} \min \{|x - \frac{1}{n}|, |x - \frac{1}{n+1}|\}$. Then $\forall x' \in (x - \delta, x + \delta)$, we have $f(x') = 0$. Thus, $\forall \varepsilon > 0, \exists \delta$ such that $\forall x' \in (x - \delta, x + \delta)$, we have $|f(x) - f(x')| = |0 - 0| = 0 < \varepsilon$. Since f is discontinuous at a countable set of points, we know the Riemann integral exists by Proposition 375 and is given by $R \int_0^1 f(x)dx = 0$. See Figure 5.2.1.3.

Example 378 Finally, let $\{q_i\}$ be the enumeration of all the rational numbers in $[0, 1]$ and let $Q_n = \{q_i \in \mathbb{Q} \cap [0, 1] : i = 1, 2, \dots, n\}, n \in \mathbb{N}$. Define, for each $n \in \mathbb{N}$, the function $f_n : [0, 1] \rightarrow \{0, 1\}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in Q_n \\ 0 & \text{otherwise} \end{cases} .$$

The function f_n is discontinuous only at the n points of Q_n in $[0, 1]$. Since f_n is continuous a.e. and is bounded, the Riemann integral exists and $R \int_0^1 f_n(x) dx = 0$. Notice however that while $f_n \rightarrow f$, $R \int_0^1 f_n(x) dx$ does not converge to $R \int_0^1 f(x) dx$ since the latter doesn't even exist!

5.2.2 Lebesgue integrals

Now that we've exposed some of the problems with the Riemann integral, we take up a systematic treatment of the Lebesgue integral. As Proposition 375 suggests, the class of Riemann integrable functions is somewhat narrow. On the other hand, the Lebesgue integrable functions are (relatively) larger. This is because the Lebesgue integral replaces the class of step functions (used in the construction of the Riemann integral) with the larger class of simple functions that were defined in 354. The essential difference between step functions and simple functions is the class of sets upon which they are defined. In particular the collection of subsets upon which the step function is defined is a strict subset of the collection of subsets upon which simple functions are defined. We will construct Lebesgue integrals under three separate assumptions concerning boundedness of the function over which we are integrating (f) and the finiteness of the measure (m) of the sets (E) upon which the function (f) is defined.

Assumption 1: f is bounded and $m(E) < \infty$

Consider the representation $\varphi : E \rightarrow \mathbb{R}$ defined in Example 355 given by $\varphi(x) = \sum_{i=1}^k a_i \chi_{A_i}(x)$ where $A_i \subset E \in \mathcal{L}$ are disjoint and $a_i \in \mathbb{R} \setminus \{0\}$ are distinct. Then we define the **elementary integral** of this simple function to be $\int \varphi(x) dx = \sum_{i=1}^k a_i m(A_i)$. This integral is well defined since $m(A_i) < \infty \forall i$ and there are finitely many terms in the sum. In this case, we call the function φ an **integrable simple function**. To economize on notation, let $\int_E \varphi \equiv \int \varphi(x) dx$.

Sometimes it is useful to employ representations that are not canonical and the following lemma asserts that the elementary integral is independent of its representation.

Lemma 379 *Let $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$, with $E_i \cap E_j = \emptyset$ for $i \neq j$. Suppose each E_i is an \mathcal{L} -measurable set of finite measure. Then $\int_E \varphi = \sum_{i=1}^n a_i m(E_i)$.*

Proof. The set $A_a = \{x \in E : \varphi(x) = a\} = \cup_{a_i=a} E_i$. Hence $amA_a = \sum_{a_i=a} a_i m(E_i)$ by additivity of m . Thus, $\int_E \varphi = \sum am(A_a) = \sum a_i m(E_i)$. ■

Next we establish two basic properties of the elementary integral.

Theorem 380 *Let φ and ψ be simple functions which vanish outside a set of measure zero.⁹ Then (i) integration preserves linearity: $\int_E (a\varphi + b\psi) = a \int_E \varphi + b \int_E \psi$ and (ii) integration preserves monotonicity: if $\varphi \geq \psi$ a.e., then $\int_E \varphi \geq \int_E \psi$.*

Exercise 5.2.1 *Prove Theorem 380.*

Let $f : E \rightarrow \mathbb{R}$ be any bounded function and E an \mathcal{L} -measurable set with $mE < \infty$. In analogy with the Riemann integral, we define the **upper Lebesgue integral** of f over E by $L^u \int_E f(x)dx = \inf_{\psi \geq f} \int_E \psi$ and the **lower Lebesgue integral** of f over E is defined by $L^l \int_E f(x)dx = \sup_{\varphi \leq f} \int_E \varphi$ where ψ and φ range over the set of all simple functions defined on E . Notice that $L^u \int_E f(x)dx$ and $L^l \int_E f(x)dx$ are well defined since f is bounded and m has finite measure on E . See Figure 5.2.2.1.

Definition 381 *If $L^u \int_E f(x)dx = L^l \int_E f(x)dx$, then we say the **Lebesgue integral** exists and denote it $\int_E f(x)dx$.*

Notice that if f is a simple function, then $\inf_{\psi \geq f} \int_E \psi = \int_E f$ and $\sup_{\varphi \leq f} \int_E \varphi = \int_E f$ so that $L^u \int_E f(x)dx = L^l \int_E f(x)dx$. Hence simple functions are Lebesgue integrable.

The next question is what other functions are Lebesgue integrable? The next theorem provides necessary and sufficient conditions for integrability. In particular, sufficiency shows that if one can establish that the function is \mathcal{L} -measurable (as well as the conditions under which this section is based), then we know it is integrable. This is another theorem like Heine-Borel where sufficiency makes one's life simple.

Theorem 382 *A bounded function f defined on an \mathcal{L} -measurable set E of finite measure is Lebesgue integrable iff f is \mathcal{L} -measurable.*

⁹We say a function f vanishes outside a set of measure zero if $m(\{x \in E : f(x) \neq 0\}) = 0$ or outside a set of finite measure if $m(\{x \in E : f(x) \neq 0\}) < \infty$.

Proof. (Sketch) (\Leftarrow) Since $f : E \rightarrow \mathbb{R}$ is bounded, $-M \leq f(x) \leq M$. Divide $[-M, M]$ into n equal parts. Construct sets

$$E_k = f^{-1} \left(\left[\frac{k-1}{n}M, \frac{k}{n}M \right) \right)$$

(i.e. E_k is the set of all $x \in E$ such that $f(x)$ belongs to a slice $[\frac{k-1}{n}M, \frac{k}{n}M)$). See Figure 5.2.2.2. E_k is measurable because it is an inverse image of a measurable function of an interval. Define two simple functions $\psi_n(x) = \frac{M}{n} \sum_{k=-n}^n k \chi_{E_k}(x)$ and $\varphi_n(x) = \frac{M}{n} \sum_{k=-n}^n (k-1) \chi_{E_k}(x)$. Then φ approximates f from below and ψ approximates f from above. Because φ and ψ are simple functions, $\int_E \varphi$ and $\int_E \psi$ are well defined. The upper (lower) Lebesgue integrals of f on E , being the infimum (supremum) satisfy

$$\int_E \varphi_n \leq L^l \int_E f \leq L^u \int_E f \leq \int_E \psi_n.$$

As n gets larger, φ_n and ψ_n get closer to each other and hence so do their integrals. Thus for $n \rightarrow \infty$, $L^l \int_E f = L^u \int_E f$ which means $\int_E f$ exists.

(\Rightarrow) Let f be integrable. Then

$$\inf_{\psi \geq f} \int \psi(x) dx = \sup_{\varphi \leq f} \int \varphi(x) dx$$

where φ and ψ are simple functions. Then by the property of infimum and supremum, for any n , there are simple functions φ_n and ψ_n such that $\varphi_n(x) \leq f(x) \leq \psi_n(x)$ and

$$\int \psi_n(x) dx - \int \varphi_n(x) dx < \frac{1}{n}. \quad (5.4)$$

Define $\psi^* = \inf_n \psi_n$ and $\varphi^* = \sup_n \varphi_n$, which are measurable by Theorem 357 and satisfy $\varphi^*(x) \leq f(x) \leq \psi^*(x)$. But the set of x for which $\varphi^*(x)$ differs from $\psi^*(x)$ (i.e. $\Delta = \{x \in E : \varphi^*(x) < \psi^*(x)\}$) has measure zero due to (5.4). Thus $\varphi^* = \psi^*$ except on a set of measure zero. Thus f is measurable by Theorem 361. ■

Notice that the assumptions on boundedness and finite measure imply $MmE < \infty$ upon which the proof rests.

Next we establish that the Lebesgue integral is a generalization of the Riemann integral.

Theorem 383 *Let f be a bounded function on $[a, b]$. If f is Riemann integrable over $[a, b]$, then it is Lebesgue integrable and $R \int_a^b f(x)dx = \int_{[a,b]} f(x)dx$.*

Proof. The proof rests on the fact that every step function (upon which Riemann integrals are defined) is also a simple functions (upon which Lebesgue integrals are defined), while the converse is not true. Then

$$R \int_a^b f(x)dx \leq \sup_{\varphi \leq f} \int_{[a,b]} \varphi(x)dx \leq \inf_{\psi \geq f} \int_{[a,b]} \psi(x)dx \leq R \int_a^b f(x)dx$$

where the first and third inequalities follow from the above fact and the second follows from the fact that $\varphi \leq f \leq \psi$ and (ii) of Theorem 380. ■

Of course, the converse is not true.

Example 384 *In Example 376 we showed that the Dirichlet function was not Riemann integrable. However, it is Lebesgue integrable by Theorem 382 since it is \mathcal{L} -measurable (which is clear since it is a simple function). Hence $\int_{[0,1]} f(x)dx = 1 \cdot m(\mathbb{Q} \cap [0, 1]) + 0 \cdot m([0, 1] \setminus \mathbb{Q}) = 1 \cdot 0 + 0 \cdot 1$.*

Now we establish the following properties of Lebesgue integrals which follow as a consequence of the fact that Lebesgue integrals are defined on simple functions and elementary integrals preserve linearity and monotonicity by Theorem 380.

Theorem 385 *If f and g are bounded \mathcal{L} -measurable functions defined on a set E of finite measure, then: (i) $\int_E (af + bg) = a \int_E f + b \int_E g$; (ii) if $f = g$ a.e., then $\int_E f = \int_E g$; (iii) if $f \leq g$ a.e., then $\int_E f \leq \int_E g$ and hence $|\int_E f| \leq \int_E |f|$; (iv) if $c \leq f(x) \leq d$, then $cm(E) \leq \int_E f \leq dm(E)$; and (v) if A and B are disjoint \mathcal{L} -measurable sets of finite measure, then $\int_{A \cup B} f = \int_A f + \int_B f$.*

Exercise 5.2.2 *Prove Theorem 385.*

We now prove a very important result concerning the interchange of limit and integral operations of a convergent sequence of bounded \mathcal{L} -measurable functions.

Theorem 386 (Bounded Convergence) *Let $\langle f_n \rangle$ be a sequence of \mathcal{L} -measurable functions defined on a set E of finite measure and suppose $|f_n(x)| \leq M$, $\forall n \in \mathbb{N}$ and $\forall x \in E$. If $f_n \rightarrow f$ a.e. on E , then f is integrable and $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$.*

Proof. (Sketch) Since $f_n \rightarrow f$, f is measurable by Theorem 364. Since f_n is uniformly bounded, then f is bounded. Given ε , it is possible to split E (by Theorem 366) into two parts $E \setminus A$ where $f_n \rightarrow f$ uniformly and A with $m(A) < \varepsilon$. Then,

$$\lim_{n \rightarrow \infty} \int_E f_n \rightarrow \int_E \lim_{n \rightarrow \infty} f_n = \int f$$

if

$$\left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - f) \right|$$

is sufficiently small. Split this integral into two parts:

$$\left| \int_E (f_n - f) \right| \leq \int_E |f_n - f| \leq \int_{E \setminus A} |f_n - f| + \int_A |f_n - f|$$

The first integral is sufficiently small because $f_n \rightarrow f$ uniformly and the second is sufficiently small because $|f_n - f|$ is bounded and $m(A)$ is sufficiently small.

■

It is important to note that $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$ only requires point-wise convergence with Lebesgue integration. A similar result for Riemann integration (i.e. $R \int_E f = \lim_{n \rightarrow \infty} R \int_E f_n$) requires uniform convergence.

Example 387 Here we return to Example 378. There we saw that the bounded function f_n was discontinuous at the n points of the \mathcal{L} -measurable set $Q_n = \{q_i \in \mathbb{Q} \cap [0, 1] : i = 1, 2, \dots, n\}$, $n \in \mathbb{N}$. While $R \int_0^1 f_n(x) dx = 0$ along the sequence and while $f_n \rightarrow f$, we saw $\lim_{n \rightarrow \infty} R \int_0^1 f_n(x) dx$ did not exist. On the other hand, since the bounded function f_n is \mathcal{L} -measurable and $m[0, 1] < \infty$, we know $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$ exists and equals 0 by Example 384.

Assumption 2: f is nonnegative and $m(E) \leq \infty$

In many instances, economists consider functions which are unbounded (e.g. most utility functions we write down are of this variety). Hence, it would be nice to relax the above assumption about boundedness. This section does that, albeit at the cost that f must be nonnegative. Here we also do not require E to be of finite measure.

Definition 388 If $f : E \rightarrow \mathbb{R}_+$ on an \mathcal{L} -measurable set E is \mathcal{L} -measurable, we define $\int_E f = \sup_{h \leq f} \int_E h$ where h is a bounded, \mathcal{L} -measurable function which vanishes outside a set of finite measure.

Notice that the integral is defined on any function h (not just simple functions) which satisfies the conditions of the previous subsection and $\sup_{h \leq f} \int_E h$ is similar to the definition of the lower Lebesgue integral in the previous subsection. That is, h is bounded and $mH = m(\{x \in E : h(x) \neq 0\}) < \infty$. Then $\int_E h$ and $\sup_{h \leq f} \int_E h$ are well defined. Furthermore $\int_E h = \int_H h + \int_{E \setminus H} h = \int_H h$.

Definition 389 A nonnegative \mathcal{L} -measurable function f defined on an \mathcal{L} -measurable set E is **integrable (or summable)** if $\int_E f < \infty$. If $\int_E f = \infty$, we say f is not integrable even though it has a Lebesgue integral.

Example 390 Let the function $f : [0, 1] \rightarrow \mathbb{R}_+$ be given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1] \\ 0 & x = 0 \end{cases}.$$

While f is unbounded, consider the sequence of functions $h_n : [0, 1] \rightarrow \mathbb{R}$ given by

$$h_n(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (\frac{1}{n}, 1] \\ n & x \in [0, \frac{1}{n}] \end{cases}.$$

In this case $h_n \leq f$ (except at $h_n(0) = n$ and $f(0) = 0$ but this is a set of measure 0) and h_n is a bounded, \mathcal{L} -measurable function which vanishes outside a set of finite measure. See Figure 5.2.2.3. Then $\int_{[0,1]} h_n(x) dx = \int_{[0, \frac{1}{n}]} n dx + \int_{(\frac{1}{n}, 1]} \frac{1}{x} dx = n \cdot (\frac{1}{n} - 0) + \ln(1) - \ln(\frac{1}{n}) = 1 + \ln(n)$. Since $\{h_n\}$ is contained in the set of all bounded h such that $h \leq f$, as we take the sup over all such functions, we know $1 + \ln(n) \rightarrow \infty$ as $n \rightarrow \infty$, so that f is not integrable on $[0, 1]$.

Exercise 5.2.3 Let the function $f : [1, \infty) \rightarrow \mathbb{R}$ be given by $f(x) = \frac{1}{x}$. Is f bounded? Is $m[1, \infty)$ finite? Is f integrable?

Lemma 391 (Chebyshev's Inequality) Let φ be an integrable function on A and $\varphi(x) \geq 0$ a.e. on A . Let $c > 0$. Then $c \cdot m\{x \in A : \varphi(x) \geq c\} \leq \int_A \varphi(x) dm$.

Proof. Let $\hat{A} = \{x \in A : \varphi(x) \geq c\}$. Then $\int_A \varphi(x) dm = \int_{\hat{A}} \varphi(x) dm + \int_{A \setminus \hat{A}} \varphi(x) dm \geq \int_{\hat{A}} \varphi(x) dm \geq cm(\hat{A})$. See Figure 5.2.2.4. ■

As in the previous subsection, there are various linearity and monotonicity properties associated with Lebesgue integrals of non-negative \mathcal{L} -measurable functions.

Theorem 392 *Let f and g be nonnegative \mathcal{L} -measurable functions defined on a set E . Then (i) $\int_E cf = c \int_E f$, $c > 0$; (ii) $\int_E (f + g) = \int_E f + \int_E g$; and (iii) if $g \geq f$ a.e., then $\int_E g \geq \int_E f$.*

Proof. (ii) Let h and k be bounded, \mathcal{L} -measurable functions such that $h \leq f$, $k \leq g$ and vanish outside sets of finite measure. Then $h + k \leq f + g$ so that $\int_E h + \int_E k = \int_E (h + k) \leq \int_E (f + g)$. Then $\sup_{h \leq f} \int_E h + \sup_{k \leq g} \int_E k \leq \int_E (f + g)$ so by Definition 388 we have (i.e. $\int_E f + \int_E g \leq \int_E (f + g)$).

To establish the reverse inequality, let l be a bounded \mathcal{L} -measurable function which vanishes outside a set of finite measure and is such that $l \leq f + g$. Define h and k by setting $h(x) = \min(f(x), l(x))$ and $k(x) = l(x) - h(x)$. Then $h \leq f$ (by construction) and $k \leq g$ also follows from $l = h + k \leq f + g$. Furthermore, h and k are bounded by the bound for l and vanish where l vanishes. Then $\int_E l = \int_E h + \int_E k \leq \int_E f + \int_E g$. But this implies $\sup_{l \leq f+g} \int_E l \leq \int_E f + \int_E g$ or $\int_E (f + g) \leq \int_E f + \int_E g$. ■

Exercise 5.2.4 *Let f be a nonnegative \mathcal{L} -measurable function. Show that $f = 0$ a.e. on E iff $\int_E f = 0$.*

As in the previous subsection, we now prove some important results concerning the interchange of limit and integral operations. The bounded convergence theorem has one restrictive assumption. It is that the sequence $\langle f_n \rangle$ is uniformly bounded. In the following lemma this assumption is dropped. Instead we assume nonnegativity of $\langle f_n \rangle$ and the result is stated in terms of inequality rather than equality.

Theorem 393 (Fatou's Lemma) *Let $\langle f_n \rangle$ be a sequence of nonnegative \mathcal{L} -measurable functions and $f_n(x) \rightarrow f(x)$ a.e. on E . Then $\int_E f \leq \underline{\lim}_{n \rightarrow \infty} \int_E f_n$.*

Proof. (Sketch) Let $\langle f_n \rangle \rightarrow f$ pointwise on E . The idea of the proof is to use the Bounded Convergence Theorem 386. To do so, we need a uniformly bounded sequence of functions. Hence, let h be a bounded function such that $h(x) \leq f(x)$, obtaining non-zero values only on a subset of E with finite measure. Define a new sequence $\langle h_n \rangle$ by $h_n(x) = \min(f_n(x), h(x))$. Then $h_n(x)$ is uniformly bounded, $h_n(x) \leq f_n(x)$ and $h_n \rightarrow h$ pointwise. Thus by the bounded convergence theorem

$$\int_E h = \lim_{n \rightarrow \infty} \int_E h_n \leq \liminf_{n \rightarrow \infty} \int_E f_n \quad (5.5)$$

where we use the *lim inf* since the limit of $\langle f_n(x) \rangle$ may not exist. Since (5.5) holds for any h with the given properties, it also holds for the supremum

$$\sup_{h \leq f} \int_E h \leq \liminf_{n \rightarrow \infty} \int_E f_n. \quad (5.6)$$

But the left hand side of (5.6) is by definition $\int_E f$. ■

The next example shows that the strict inequality may be obtained.

Example 394 Let the functions $f_n : [0, 1] \rightarrow \mathbb{R}_+$ be given by

$$f_n(x) = \begin{cases} n & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

See Figure 5.2.2.5. In this case $\lim_{n \rightarrow \infty} f_n(x) = 0$ a.e. and $\underline{\lim}_{n \rightarrow \infty} \int_{[0,1]} f_n(x) dx = \sup_{n \rightarrow \infty} \left[\inf_{k \geq n} \int_{[0,1]} f_k(x) dx \right] = \sup\{\langle 1 \rangle\} = 1$.¹⁰

To see that nonnegativity matters for Fatou's lemma, consider the following example.

Example 395 Instead, let the functions $f_n : [0, 1] \rightarrow \mathbb{R}_+$ be given by

$$f_n(x) = \begin{cases} -n & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Again $\lim_{n \rightarrow \infty} f_n(x) = 0$ a.e. and $\underline{\lim}_{n \rightarrow \infty} \int_{[0,1]} f_n(x) dx = \sup_{n \rightarrow \infty} \left[\inf_{k \geq n} \int_{[0,1]} f_k(x) dx \right] = \sup\{-1, -2, -3, \dots\} = -1$. Hence without nonnegativity we may have $\int_E f > \underline{\lim}_{n \rightarrow \infty} \int_E f_n$.

The conclusion of Theorem 393 is weak. It is possible to strengthen it by imposing more structure on the sequence of functions.

Theorem 396 (Monotone Convergence) Let $\langle f_n \rangle$ be an increasing sequence of nonnegative \mathcal{L} -measurable functions and $f_n(x) \rightarrow f(x)$ a.e. on E . Then $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$.

Proof. Since $f_n \leq f, \forall n$ we have $\int_E f_n \leq \int_E f$ by (iii) of Theorem 392. This implies $\overline{\lim}_{n \rightarrow \infty} \int_E f_n \leq \int_E f$. The result then follows from Theorem 393. ■

¹⁰In this example, it was not necessary to actually take the liminf

Example 397 Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \in (0, 1] \\ 0 & x = 0 \end{cases} .$$

As in Example 390, while f is unbounded, consider a sequence of functions $h_n : [0, 1] \rightarrow \mathbb{R}$ given by

$$h_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n \end{cases} .$$

or in other words

$$h_n(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \in [\frac{1}{n^2}, 1] \\ n & x \in [0, \frac{1}{n^2}) \end{cases} .$$

In this case $h_n \leq f$ (except at $x = 0$ but this is a set of measure 0) and h_n is a bounded, \mathcal{L} -measurable function which vanishes outside a set of finite measure. Furthermore, h_n is monotone since $h_n(x) \leq h_{n+1}(x), \forall x \in [0, 1]$. Then $\int_{[0,1]} h_n(x) dx = \int_{[0, \frac{1}{n^2}]} n dx + \int_{(\frac{1}{n^2}, 1]} \frac{1}{\sqrt{x}} dx = n \cdot (\frac{1}{n^2} - 0) + 2(1 - \frac{1}{n}) = 2 - \frac{1}{n}$. Then as $n \rightarrow \infty$, $\int_{[0,1]} h_n(x) dx = 2$. By the Monotone Convergence Theorem, $\int_{[0,1]} f(x) = 2$.

Example 397 is known as an improper integral when regarded as a Riemann integral since the integrand is unbounded.¹¹ On the other hand, it is perfectly proper when regarded as a Lebesgue integral. In this example, the two integrals are equal. Furthermore, we note that while Example 397 provides a case in which an unbounded nonnegative \mathcal{L} -measurable function is integrable, Example 390 provides an instance of a closely related function which is not integrable.

Assumption 3: f is any function and $m(E) \leq \infty$ (general lebesgue integral)

Definition 398 An \mathcal{L} -measurable function f is **integrable** over E if f^+ and f^- are both integrable over E . In this case, $\int_E f = \int_E f^+ - \int_E f^-$.

Theorem 399 A function f is integrable over E iff $|f|$ is integrable over E .

¹¹We also say that a Riemann integral is improper if its interval of integration is unbounded.

Proof. (\Rightarrow) If f is integrable over E , then f^+ and f^- are both integrable over E . Thus, $\int_E |f| = \int_E f^+ + \int_E f^-$ by Theorem 392. Hence $|f|$ is integrable.

(\Leftarrow) If $\int_E |f| < \infty$, then so are $\int_E f^+$ and $\int_E f^-$. ■

Example 400 Consider a version of the Dirichlet function $f : [0, 1] \rightarrow \{-1, 1\}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ -1 & \text{otherwise} \end{cases}.$$

Observe that $|f| = 1$ and hence Riemann integrable while f is not.

Lemma 401 Let $A \in \mathcal{L}$, $m(A) = 0$, and f be an \mathcal{L} -measurable function. Then $\int_A f = 0$.

Proof. We show it first for a simple function. Let $f = \sum_{i=1}^n \chi_{E_i}$ where $\{E_i\}$ is a collection of \mathcal{L} -measurable sets that are disjoint. Then $f\chi_A = \sum_{i=1}^n \alpha_i \chi_{A \cap E_i}$, where $A \cap E_i$ are disjoint and $m(A \cap E_i) = 0$ (since $m(A \cap E_i) \leq m(A) = 0$). Thus $\int f\chi_A = \sum_{i=1}^n \alpha_i m(A \cap E_i) = 0$.

If f is a non-negative measurable function, then by Theorem 367 there is a non-decreasing sequence $\langle f_n \rangle$ of simple functions that converges pointwise to f . Then by Theorem 396

$$\int_A f = \lim_{n \rightarrow \infty} \int_A f_n = \lim_{n \rightarrow \infty} \int f\chi_A = 0.$$

Finally, if f is an arbitrary measurable function, then $f\chi_A = f^+\chi_A - f^-\chi_A$ and $\int_A f = \int_A f^+ - \int_A f^- = 0 - 0 = 0$. ■

Lemma 402 Let f be an \mathcal{L} -measurable function over E . If there is an integrable function g such that $|f| \leq g$, then f is integrable over E .

Proof. From $f^+ \leq g$, it follows that $\int_E f^+ \leq \int_E g$, and so f^+ is integrable on E . Similarly, $f^- \leq g$ implies integrability of f^- . Hence f is integrable over E . ■

Theorem 403 Let f and g be integrable functions defined on a set E . Then
 (i) the function cf where c is finite is integrable over E and $\int_E cf = c \int_E f$;
 (ii) the function $f + g$ is integrable over E and $\int_E (f + g) = \int_E f + \int_E g$;
 (iii) if $g = f$ a.e., then $\int_E g = \int_E f$;
 (iv) if $g \geq f$ a.e., then $\int_E g \geq \int_E f$; and
 (v) If E_1 and E_2 are disjoint \mathcal{L} -measurable sets in E , then $\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$.

Exercise 5.2.5 Prove Theorem 403.

In considering when we could interchange limits and integrals, we saw either we had to impose bounds on functions (Bounded Convergence Theorem 386) or consider monotone sequences of nonnegative functions (Monotone Convergence Theorem 396). In the general case, we simply must bound the sequence of functions by another (possibly unbounded) function.

Theorem 404 (Lebesgue Dominated Convergence Theorem) Let g be an integrable function on E and let $\langle f_n \rangle$ be a sequence of \mathcal{L} -measurable functions such that $|f_n| \leq g$ on E and $\lim_{n \rightarrow \infty} f_n = f$ a.e. on E . Then $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$.

Proof. (Sketch) By Lemma 402, f is integrable. We want to use Fatou's Lemma 393 which requires a sequence of non-negative functions, which is not assumed in this theorem. However we can define two sequences, namely $h_n = f_n + g$ and $k_n = g - f_n$ for which $\langle h_n \rangle \rightarrow f + g$ and $\langle k_n \rangle \rightarrow g - f$ where both are non-negative. Hence by Fatou's Lemma, we have $\int_E (f + g) \leq \liminf_{n \rightarrow \infty} \int_E (f_n + g)$ and $\int_E (g - f) \leq \liminf_{n \rightarrow \infty} \int_E (g - f_n)$. The first inequality implies $\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$ and the second implies $\int f \geq \limsup_{n \rightarrow \infty} \int f_n$ by Theorem 403. Combining these two we have

$$\liminf_{n \rightarrow \infty} \int_E f_n \geq \int f \geq \limsup_{n \rightarrow \infty} \int f_n$$

which implies the desired result. ■

The above theorem requires that the sequence $\langle f_n \rangle$ be uniformly dominated by a fixed integrable function g . However, the proof does not need such a strong restriction. In fact, the requirements can be weakened to consider a sequence of integrable functions $\langle g_n \rangle$ which converge a.e. to an integrable function g and that $|f_n| \leq g_n$.

Example 405 Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be given by $f_n(x) = nx^n$. See Figure 5.2.2.6. Then $\lim_{n \rightarrow \infty} f_n(x) = 0$ a.e. and $\int_{[0,1]} f_n(x) dx = \int_{[0,1]} nx^n dx = \frac{n}{n+1} x^{n+1} \Big|_0^1 = \frac{1}{1+\frac{1}{n}}$ so that $\lim_{n \rightarrow \infty} \int_{[0,1]} f_n(x) dx = 1$. On the other hand, $\int_{[0,1]} f(x) dx = 0$.

Notice in the above example that the sequence of functions has no dominating function.

Example 406 Let $f_n : [0, 2] \rightarrow \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} \sqrt{n} & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

See Figure 5.2.2.7. Then $\lim_{n \rightarrow \infty} f_n(x) = f(x) = 0 \forall x \in [0, 2]$ so that $\int_{[\frac{1}{n}, \frac{2}{n}]} f(x) dx = 0$. Note that $|f_n(x)| \leq g(x) \forall x \in [0, 2]$ where

$$g(x) = \begin{cases} \sqrt{\frac{2}{x}} & \text{if } 0 < x \leq 2 \\ 0 & \text{if } x = 0 \end{cases}$$

which is integrable over $[0, 2]$. It is also simple to see $\int_{[\frac{1}{n}, \frac{2}{n}]} f_n(x) dx = \sqrt{n} (\frac{2}{n} - \frac{1}{n}) = \frac{1}{\sqrt{n}}$ so that $\lim_{n \rightarrow \infty} \int_{[\frac{1}{n}, \frac{2}{n}]} f_n(x) dx = 0$.

Finally we state a convergence theorem that differs from the previous ones in the sense that we don't assume that $\langle f_n \rangle \rightarrow f$. Instead the theorem guarantees the existence of a function f to which $\langle f_n \rangle$ converges a.e. given $\langle f_n \rangle$ is a non-decreasing sequence of integrable functions with corresponding sequence of their integrals $\langle \int f_n dm \rangle$ being bounded.

Theorem 407 (Levi) Let $\langle f_n \rangle$ be a sequence on $A \subset \mathbb{R}$ and $f_1 \leq f_2 \leq \dots \leq f_n \leq \dots$ where f_n is integrable and $\int_A f_n dm \leq K$. Then there exists f s.t. $f = \lim_{n \rightarrow \infty} f_n$ a.e. on A , f is integrable on A and $\int_A f_n dm \rightarrow \int_A f dm$.

Proof. (Sketch) Without loss of generality, assume $f_1 \geq 0$. Define $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Since $\langle f_n \rangle$ is non-decreasing, $f(x)$ is either a number or $+\infty$. Using the Chebyshev's inequality (Lemma 391), it is easy to show that $m(\{x \in A : f(x) = +\infty\}) = 0$, which implies $f_n \rightarrow f$ pointwise a.e. In order to use the Lebesgue Dominated Convergence Theorem 404, we need to construct an integrable function φ on A that dominates f_n (i.e. $f_n \leq f \leq \varphi$ on A). See Figure 5.2.2.8. Let $A_r = \{x \in A : r-1 \leq f(x) < r\}$ and $\varphi(x) = \sum_{r=1}^{\infty} r \chi_{A_r}$. Clearly $f_n \leq f(x) \leq \varphi$. Is φ integrable on $A = \cup_{r=1}^{\infty} A_r$ (i.e. is $\int_A \varphi = \sum_{r=1}^{\infty} r m(A_r) < \infty$)? For $s \in \mathbb{N}$, define $B_s = \cup_{r=1}^s A_r$. Since $\varphi(x) \leq f(x) + 1$ and both f_n and f are bounded on B_s , we have

$$\sum_{r=1}^s r m(A_r) = \int_{B_s} \varphi dm \leq \int_{B_s} f(x) dm + m(A) = \lim_{n \rightarrow \infty} \int_{B_s} f_n(x) dm + m(A) \leq K + m(A).$$

Boundedness of partial sums of an infinite series $\sum_{r=1}^{\infty} r m(A_r)$ ($= \int_A \varphi dm$) guarantees integrability of φ . ■

We can also state a “series version” of Levi’s theorem. It says that under certain conditions on the series, integration and infinite summation are interchangeable.

Corollary 408 *If $\langle g_k \rangle$ is a sequence of non-negative functions defined on A such that $\sum_{k=1}^{\infty} \int_A g_k(x) dm < \infty$, then the infinite series $\sum_{k=1}^{\infty} g_k(x)$ converges a.e. on A . That is, $\sum_{k=1}^{\infty} g_k(x) \rightarrow g(x)$ a.e. and $\sum_{k=1}^{\infty} \int_A g_k(x) dm = \int_A \sum_{k=1}^{\infty} g_k(x) dm$ ($= \int_A g(x) dm$).*

Proof. Apply Levi’s Theorem to the functions $f_n(x) = \sum_{k=1}^n g_k(x)$. ■

5.3 General Measure

In the preceding sections, we focussed on \mathbb{R} (or subsets thereof) as the underlying set of interest. From this set, we constructed the Lebesgue σ -algebra denoted \mathcal{L} . Then we defined the Lebesgue measure m on elements of \mathcal{L} . That is, we studied the triple $(\mathbb{R}, \mathcal{L}, m)$ known as the Lebesgue measure space. These ideas can be extended to general measure spaces.

Definition 409 *The pair (X, \mathcal{X}) , where X is any set and \mathcal{X} is a σ -algebra of its subsets is called a **measurable space**. Any set $A \in \mathcal{X}$ is called a **(\mathcal{X} -)measurable set**.*

Definition 410 *Let (X, \mathcal{X}) be a measurable space. A **measure** is an extended real valued function $\mu : \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$ such that: (i) $\mu(\emptyset) = 0$; (ii) $\mu(A) \geq 0, \forall A \in \mathcal{X}$; and (iii) μ is countably additive (i.e. if $\{A_n\}_{n \in \mathbb{N}}$ is a countable, disjoint sequence of subsets $A_n \in \mathcal{X}$, then $\mu(\cup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$).*

Definition 411 *A **measure space** is a triple (X, \mathcal{X}, μ) .*

Definition 412 *Let (X, \mathcal{X}) be a measurable space. A measure μ is called **finite** if $\mu(X) < \infty$. μ is called **σ -finite** if there is a countable collection of sets $\{E_i\}_{i=1}^{\infty}$ in \mathcal{X} with $\mu(E_i) < \infty$ for all i and $X = \cup_{i=1}^{\infty} E_i$.*

Example 413 (i) If $X = \mathbb{R}$, then Lebesgue measure is not finite because $m(-\infty, \infty) = \infty$ but it is σ -finite because $(-\infty, \infty) = \cup_{n=1}^{\infty} (-n, n)$ and $m((-n, n)) = 2n$. (ii) If $X = [0, 100]$, then Lebesgue measure is finite because $m([0, 100]) = 100$.

Exercise 5.3.1 Let $X = (-\infty, \infty)$, $\mathcal{X} = \mathcal{P}(X)$ and $\mu(E) = \begin{cases} \# \text{ of elements} & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is infinite} \end{cases}$. Show that (X, \mathcal{X}, μ) is a measure space and that μ is not σ -finite.

Lebesgue measure has one important property that follows from Exercise 5.1.2. That is, if E is a Lebesgue measurable set with measure 0 and if $A \subset E$, then A is also Lebesgue measurable and $m(A) = 0$. In general, however, we can have a situation that $E \in \mathcal{X}$ (is \mathcal{X} -measurable) with $\mu(E) = 0$, $A \subset E$ but A may not be in \mathcal{X} .

Example 414 Let (X, \mathcal{X}, μ) be a measure space defined as follows: Let $X = \{a, b, c\}$, $\mathcal{X} = \{\emptyset, \{a, b, c\}, \{a, b\}, \{c\}\}$; $\mu(\emptyset) = \mu(\{a, b\}) = 0$, $\mu(\{c\}) = \mu(\{a, b, c\}) = 1$, $\{a\} \subset \{a, b\}$ but $\{a\}$ is not \mathcal{X} -measurable.

Definition 415 Let (X, \mathcal{X}, μ) be a measure space. μ is **complete** on \mathcal{X} if for any $E \in \mathcal{X}$ with $\mu(E) = 0$ and for $A \subset E$ then $A \in \mathcal{X}$ and $\mu(A) = 0$.

That is, μ is complete in \mathcal{X} if any subset of a zero measurable set is measurable and has measure zero.

If we consider Lebesgue measure restricted to the Borel σ -algebra (i.e. $(\mathbb{R}, \mathcal{B}, m)$) then m is not complete on \mathcal{B} . To show this would necessitate more machinery (the Cantor set can be used to illustrate the idea). However, if μ is not complete on \mathcal{X} , then there exists a completion of \mathcal{X} denoted by $\tilde{\mathcal{X}}$. For example, the completion of $(\mathbb{R}, \mathcal{B}, m)$ is $(\mathbb{R}, \mathcal{L}, m)$.

Exactly the same way we built the theory of Lebesgue measure and Lebesgue integral in Sections 5.1 to 5.2.2, the theory of general measure and integral can be constructed. The space $(\mathbb{R}, \mathcal{L}, m)$ can be replaced by (X, \mathcal{X}, m) and instead of \mathcal{L} -measurability and \mathcal{L} -integrability we will have \mathcal{X} -measurability and \mathcal{X} -integrability.

5.3.1 Signed Measures

Although we have introduced a general measure space (X, \mathcal{X}, μ) , the only non-trivial measure space that we have encountered so far is the Lebesgue

measure space $(\mathbb{R}, \mathcal{L}, m)$, where Lebesgue measure m was constructed through the outer measure in Section 5.1. Can we construct other non-trivial measures μ on a general measure space (X, \mathcal{X}) ? Consider a measure space (X, \mathcal{X}, μ) and let f be a non-negative \mathcal{X} -measurable function. Define $\lambda : \mathcal{X} \rightarrow \mathbb{R}$ by $\lambda(E) = \int_E f d\mu$. Then the following theorem establishes that this set function λ is a measure.

Theorem 416 *If f is a non-negative \mathcal{X} -measurable function then*

$$\lambda(E) = \int_E f d\mu \tag{5.7}$$

is a measure. Moreover if f is \mathcal{X} -integrable, then λ is finite.

Proof. $\lambda(E) = \int_E f d\mu = \int_X f \chi_E d\mu \geq 0$ for all $E \in \mathcal{X}$ (where χ_E is the characteristic function of E). Let $\{E_i\}_{i=1}^\infty$ be a collection of mutually disjoint sets and $\cup_{i=1}^\infty E_i = E$. Then $\chi_E = \sum_{i=1}^\infty \chi_{E_i}$. Let $g_n = f \chi_{E_n}$ and $f_n = \sum_{i=1}^n g_n$. Since $g_n \geq 0$, the sequence $\langle f_n \rangle$ is non-decreasing. f_n is also measurable for all n because the sum and the product of measurable functions is measurable. f_n converges pointwise to f because $\sum_{i=1}^n \chi_{E_i} \rightarrow \chi_E = \sum_{i=1}^\infty \chi_{E_i}$. Then according to the monotone convergence theorem 396

$$\lambda(E) = \int_E f d\mu = \int f \chi_E d\mu = \int f \sum_{i=1}^\infty \chi_{E_i} = \sum_{i=1}^\infty \int f \chi_{E_i} d\mu = \sum_{i=1}^\infty \int_{E_i} f d\mu = \sum_{i=1}^\infty \lambda(E_i)$$

Hence λ is σ -additive. If f is integrable, then $\lambda(X) = \int_X f d\mu < \infty$ and hence f is finite. ■

This theorem provides us with a method of how to construct new measures on a measure space (X, \mathcal{X}, μ) . Actually any non-negative \mathcal{X} -integrable function represents a finite measure given by 5.7. Thus, given a measure space (X, \mathcal{X}, μ) , there is a whole set of measures defined on \mathcal{X} .

Are all measures on (X, \mathcal{X}, μ) of the type given by 5.7? In other words, let \mathcal{E} be the set of all measures on (X, \mathcal{X}, μ) . Can any measure $\nu \in \mathcal{E}$ be represented by an integrable function g such that $\nu(E) = \int_E g d\mu$? The answer is contained in a well-known result: the Radon-Nikodym theorem. We could pursue this problem in our current setting; namely we could deal with measures only (i.e. with non-negative σ -additive set functions defined on \mathcal{X}). We can, however, work in an even more general setting. Instead of dealing with non-negative σ -additive set functions (measures) we can drop the assumption

of non-negativity and work with the σ -additive set functions which are either positive or negative or both. These functions are called *signed measures*. This generalization is useful particularly when working with Markov processes. ???NEED AN EXAMPLE WITH A MARKOV PROCESS HERELet us now define the notion of signed measure rigorously.

Definition 417 Let (X, \mathcal{X}) be a measurable space. Let $\mu : \mathcal{X} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ with the following properties: (i) $\mu(\emptyset) = 0$; (ii) μ obtains at most one of the two symbols $+\infty, -\infty$; (iii) μ is σ -additive. Then μ is called a **signed measure** on \mathcal{X} .

In the text that follows when we refer to a "measure" (without the prefix "signed") we mean measure in the sense of Definition 410(i.e. a non-negative set function).

Example 418 Given a measure space (X, \mathcal{X}, μ) , an example of a signed measure is the set function

$$\nu(E) = \int_E f d\mu \quad (5.8)$$

where f is any \mathcal{X} -integrable function. In Theorem 416 we showed that if f is a non-negative integrable function then P is a finite measure. Here we just assume that f is integrable and put no restrictions on non-negativity. We can also assume that f is the "only" measurable function for which $\int f d\mu$ exists (i.e. at least one of the functions f^+, f^- is integrable). Thus, if f in (5.8) is an integrable function, then ν is a finite signed measure and if f is a \mathcal{X} -measurable function for which $\int f d\mu$ exists, then ν is a signed measure (though not necessarily finite).

Example 419 Let $(\mathbb{R}, \mathcal{L}, m)$ be a Lebesgue measure space. In the first case,

let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \frac{x}{(1+x^2)^2}$. Then we have $f^+(x) = \begin{cases} \frac{x}{(1+x^2)^2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$,

$f^-(x) = \begin{cases} 0, & x \geq 0 \\ -\frac{x}{(1+x^2)^2}, & x < 0 \end{cases}$, $\int_{-\infty}^{\infty} f^+ dm = \int_{-\infty}^{\infty} f^- dm = \frac{1}{2}$, f is integrable

and $\nu(E) = \int_E f dm = \int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2} \cdot \chi_E(x) dm$ is a finite signed measure.

Example 420 Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g(x) = \begin{cases} x, & x \geq 0 \\ \frac{x}{(1+x^2)^2}, & x < 0 \end{cases}$. Then

we have $g^+(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $g^-(x) = \begin{cases} 0, & x \geq 0 \\ -\frac{x}{(1+x^2)^2}, & x < 0 \end{cases}$, $\int_{-\infty}^{\infty} g^+ dm = +\infty$, $\int_{-\infty}^{\infty} g^- dm = \frac{1}{2}$, g is measurable but not integrable. However, the integral exists since $\int_{-\infty}^{\infty} g dm = \int_{-\infty}^{\infty} g^+ dm - \int_{-\infty}^{\infty} g^- dm = +\infty - \frac{1}{2} = +\infty$, $\nu(E) = \int_E g \cdot dm = \int_{-\infty}^{\infty} g \cdot \chi_E dm$ is a signed measure (but not finite).

Example 421 Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be given by $h(x) = x$. Then we have $h^+(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $h^-(x) = \begin{cases} 0, & x \geq 0 \\ -x, & x < 0 \end{cases}$, $\int_{-\infty}^{\infty} h^+ dm = +\infty$, $\int_{-\infty}^{\infty} h^- dm = +\infty$. Hence $\int_{-\infty}^{\infty} h dm = \int_{-\infty}^{\infty} h^+ dm - \int_{-\infty}^{\infty} h^- dm = \infty - \infty$ is not defined, the Lebesgue integral doesn't exist, and thus this function doesn't define a signed measure.

The previous examples show that if a signed measure ν is defined by expression (5.8) using an integral then it can be written as a difference of two measures:

$$\begin{aligned} \nu(E) &= \nu_1(E) - \nu_2(E) \text{ where} \\ \nu(E) &= \int_E f d\mu, \nu_1(E) = \int_E f^+ d\mu \text{ and } \nu_2(E) = \int_E f^- d\mu \end{aligned}$$

We now show that such a decomposition is possible for any arbitrary signed measure. This decomposition is known as the Jordan decomposition of a signed measure. First we need to prove some lemmas. However, in order to avoid introducing complicated terminology which does not help to understand the main ideas, in the remainder of this chapter we will deal only with finite signed measures. All theorems and proofs can be adopted to σ -finite signed measures.

Lemma 422 Let ν be a finite signed measure on \mathcal{X} . Then any collection of disjoint sets $\{E_i\}$, for which $\nu(E_i) > 0$ ($\nu(E_i) < 0$) is countable.

Proof. Let $\mathcal{E} \subset \mathcal{X}$ be a collection of disjoint sets $\{E_i\}$ for which $\nu(E_i) > 0$. For $n = 1, 2, \dots$ let $\mathcal{E}_n = \{E_i \in \mathcal{E} : \nu(E_i) > \frac{1}{n}\}$. Then $\mathcal{E} = \cup_{n=1}^{\infty} \mathcal{E}_n$. For each n , \mathcal{E}_n is finite. If it were not, we would have a sequence $\langle E_k \rangle_{k=1}^{\infty}$ of disjoint sets from \mathcal{E} with $\nu(E_k) > \frac{1}{n}$ for $k = 1, 2, \dots$. Then $\nu(\cup_{k=1}^{\infty} E_k) = \sum_{i=1}^{\infty} \nu(E_k) \geq$

$\sum_{k=1}^{\infty} \frac{1}{n} = \infty$, which leads to a contradiction, that ν is finite. Then \mathcal{E}_n is finite and hence $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ is countable. ■

Let ν be a signed measure on \mathcal{X} and let $\nu(E) > 0$. Let $F \subset E$. What can be said about the sign of $\nu(F)$? As the next example shows, not much can be said about the signed measure of a subset of a set whose signed measure is positive.

Example 423 Let $X = \{1, 2, 3, \dots\}$, $\mathcal{X} = \mathcal{P}(X)$. For $E \in \mathcal{X}$, define $\nu(E) = \sum_{n \in E} (-1)^n \frac{1}{2^n}$. For $E = \{1, 2, 3\}$, $\nu(E) = -\frac{3}{8} < 0$. If $F = \{2\} \subset E$, $\nu(F) = \frac{1}{4} > 0$. But notice that each singleton subset C of set $B = \{1, 3, 5, \dots\}$ has $\nu(C) < 0$ and each singleton subset D of the set $A = \{2, 4, 6, \dots\}$ has $\nu(D) > 0$. Moreover A and B are disjoint and $A \cup B = X$.

Example 424 Let $X = [-1, 1]$ and \mathcal{X} be all \mathcal{L} -measurable subsets of $[-1, 1]$. For $E \subset [-1, 1]$, let $\nu(E) = \int_E x dx$. For $E = [-\frac{1}{2}, 1]$, then $\nu(E) = \int_{-\frac{1}{2}}^1 x dx = \frac{3}{8}$. For $F = [-\frac{1}{2}, 0] \subset [-\frac{1}{2}, 1]$, $\nu(F) = \int_{-\frac{1}{2}}^0 x dx = -\frac{1}{8}$. Thus the sign measure of the set E is positive but its subset F has negative sign measure. But for each subset C of the set $B = [-1, 0)$, $\nu(C) < 0$ and for each subset D of the set $A = [0, 1]$, $\nu(D) > 0$ where $A \cap B = \emptyset$ and $A \cup B = X$.

Definition 425 A \mathcal{X} -measurable set E is **positive (negative)** with respect to a signed measure ν , if for any \mathcal{X} -measurable subset F of E , $\nu(F) \geq 0$, ($\nu(F) \leq 0$).

Thus sets A in Examples 423 and 424 are positive while B are negative. Notice that $\nu(G) > 0$ (or $\nu(G) < 0$) doesn't mean that G is positive (negative) as E in Examples 423 and 424 show. We will show that the existence of sets A and B for a signed measure D in these examples is not a coincidence.

Definition 426 An ordered pair (A, B) , where A is a positive and B is a negative set, with respect to a signed measure ν and $A \cap B = \emptyset$, $A \cup B = X$ is called the **Hahn decomposition** with respect to ν of a measurable space (X, \mathcal{X}) .

Theorem 427 (Hahn Decomposition) Let ν be a finite signed measure on a measurable space (X, \mathcal{X}) . Then there exists a Hahn decomposition of (X, \mathcal{X}) .

Proof. (Sketch) Let \mathcal{S} be a family of collections of subsets of X whose elements \mathcal{A} are collections of disjoint measurable sets $E \subset X$ with $\nu(E) < 0$. Since “ \subset ” is a partial ordering on \mathcal{S} satisfying the assumptions of Zorn’s lemma 46 (namely that every totally ordered subcollection $\{\mathcal{A}_i\}$ has a maximal element $\mathcal{A} = \cup_i \mathcal{A}_i$), then there is a maximal element \mathcal{E} of \mathcal{S} . Moreover \mathcal{E} is countable (by Lemma 422). Let $B = \cup\{E \in \mathcal{E}\}$. Then B is measurable and negative (by construction all of its subsets have negative measure). Let $A = X \setminus B$. We have $A \cap B = \emptyset$, $A \cup B = X$, and A is measurable. If we show that A is positive, then we would be done since (A, B) would be a Hahn decomposition. Hence we need to show that A (as a complement of a maximal negative set) is positive. The idea is that if we assume that A is not positive, then we can construct a negative set with negative measure outside the set B . This would violate the maximality of B . While the construction of such a set is given in the formal proof of this theorem in the appendix to the chapter, see Figure 5.3.1. ■

In the special case where a signed measure ν is defined by the integral $\nu(E) = \int_E f d\mu$, the Hahn decomposition is given by $A = \{x : f(x) \geq 0\}$ and $B = \{x : f(x) < 0\}$ as we have seen in Example 424. It is easily seen that the Hahn decomposition is not unique. We can, for example, set $A_1 = \{x : f(x) > 0\}$, $B_1 = \{x : f(x) \leq 0\}$. But the following theorem shows that the choice of a Hahn decomposition doesn’t really matter.

Theorem 428 *Let (A_1, B_1) , (A_2, B_2) be two Hahn decompositions of a measurable space (X, \mathcal{X}) with respect to a signed measure ν . Then for each $E \in \mathcal{X}$ we have $\nu(E \cap A_1) = \nu(E \cap A_2)$ and $\nu(E \cap B_1) = \nu(E \cap B_2)$.*

Proof. From $E \cap (A_1 \setminus A_2) \subset E \cap A_1$ we have $\nu(E \cap (A_1 \setminus A_2)) \geq 0$ and from $E \cap (A_1 \setminus A_2) \subset E \cap B_2$ we have $\nu(E \cap (A_1 \setminus A_2)) \leq 0$. Combining these two inequalities we have $\nu(E \cap (A_1 \setminus A_2)) = 0$. Analogously we can show that $\nu(E \cap (A_2 \setminus A_1)) = 0$. Hence $\nu(E \cap A_1) = \nu((E \cap (A_2 \setminus A_1)) \cup (E \cap (A_1 \cap A_2))) = 0 + \nu(E \cap A_1 \cap A_2)$. If we start with $\nu(E \cap A_2)$, we arrive by similar reasoning with $0 + \nu(E \cap A_1 \cap A_2)$. Hence, $\nu(E \cap A_1) = \nu(E \cap A_2)$. Similarly we can show that $\nu(E \cap B_1) = \nu(E \cap B_2)$. ■

Theorem 429 *Let ν be a finite signed measure on \mathcal{X} and let (A, B) be an arbitrary Hahn decomposition with respect to ν . Then $\nu(E) = \nu^+(E) - \nu^-(E)$ for any $E \in \mathcal{X}$ where $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$ are both measures on \mathcal{X} and don’t depend on the choice of Hahn decomposition (A, B) .*

Proof. The independence of ν^+ and ν^- on the choice of Hahn decomposition follows from Theorem 428. Since ν is a signed measure, ν is σ -additive and thus ν^+ and ν^- are as well. Since A is a positive set and $E \cap A \subset A$, then $\nu(E \cap A) \geq 0$. Since B is a negative set and $E \cap B \subset B$, then $\nu(E \cap B) \leq 0$ so that $-\nu(E \cap B) \geq 0$. Thus ν^+ and ν^- are measures on \mathcal{X} . Since $E = E \cap X = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$, we have $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu^+(E) - \nu^-(E)$. ■

Definition 430 $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition** of a signed measure ν . The measure ν^+ (ν^-) is called a **positive (negative) variation** of ν . $|\nu|(E) = \nu^+(E) + \nu^-(E)$ is also a measure on \mathcal{X} and is called the **total variation** of a signed measure ν .

Exercise 5.3.2 Let (X, \mathcal{X}, μ) be a measure space and let f be \mathcal{X} -integrable. If $\nu(E) = \int_E f d\mu$, show that $\nu^+(E) = \int_E f^+ d\mu$, $|\nu|(E) = \int_E |f| d\mu$.

Exercise 5.3.3 Show that a countable union of positive (negative) sets is a positive (negative) set.

If a signed measure ν is defined as the integral of an integrable function $\nu(E) = \int_E f d\mu$ then by Lemma 401 it has the following property. If $E \in \mathcal{X}$ and $\mu(E) = 0$, then $\nu(E) = 0$. As we will soon see, this property of a signed measure is very important and we formulate it for any signed measure (not only the one given by an integral).

Definition 431 Let ν be a finite signed measure and let μ be a measure on (X, \mathcal{X}) . If for every $A \in \mathcal{X}$, $\mu(A) = 0$ implies $\nu(A) = 0$, then we say that ν is **absolutely continuous** with respect to μ , written $\nu \ll \mu$.

Hence by Lemma 401, $\nu(E) = \int_E f d\mu$ is absolutely continuous with respect to μ . Now we prove two simple lemmas.

Lemma 432 Let ν be a finite signed measure and μ be a measure on \mathcal{X} . Then the following are equivalent: (i) $\nu \ll \mu$, (ii) $\nu^+ \ll \mu$, $\nu^- \ll \mu$, (iii) $|\nu| \ll \mu$.

Proof. (i) \implies (ii). Let (A, B) be a Hahn decomposition with respect to ν . Let $E \in \mathcal{X}$ and $\mu(E) = 0$. Then $\mu(E \cap A) = 0$ and because ν is absolutely continuous with respect to μ , we have $\nu(E \cap A) = \nu^+(E) = 0$. This implies

$\nu^+ \ll \mu$. Similarly $\nu^- \ll \mu$. The other two implications follow immediately from these equalities:

$$\begin{aligned} |\nu|(E) &= \nu^+(E) + \nu^-(E) \\ \nu(E) &= \nu^+(E) - \nu^-(E). \end{aligned}$$

■

Lemma 433 *Let μ, λ be finite measures on \mathcal{X} , $\lambda \ll \mu$ and $\lambda(E_0) \neq 0$ for at least one set $E_0 \in \mathcal{X}$. Then there exists $\varepsilon > 0$ and a set $E \in \mathcal{X}$ that is positive with respect to the signed measure $\lambda - \varepsilon\mu$ and $\lambda(E) > 0$ and $\mu(E) > 0$.*

Proof. Let (A_n, B_n) for $n \in \mathbb{N}$ be a Hahn decomposition of X with respect to $\lambda - \frac{1}{n}\mu$. Set $A = \cup_{n=1}^{\infty} A_n$, $B = \cap_{n=1}^{\infty} B_n$. Since $B \subset B_n$ and B_n is a negative set with respect to $\lambda - \frac{1}{n}\mu$ then $(\lambda - \frac{1}{n}\mu)(B) \leq 0 \iff 0 \leq \lambda(B) \leq \frac{1}{n}\mu(B)$, for $n \in \mathbb{N}$. Thus $\lambda(B) = 0$. Since $\lambda(X) \neq 0$, then $\lambda(A) = \lambda(X \setminus B) = \lambda(X) - \lambda(B) = \lambda(X) > 0$. As $\lambda \ll \mu$ we have $\mu(A) > 0$. Finally set $E = A_{n_0}$ and $\varepsilon = \frac{1}{n_0}$. ■

Now we are ready to tackle the main problem of this section, which you can think of as a representation theorem.¹² Given a measure space (X, \mathcal{X}, μ) , consider the set function $\nu(E) = \int_E f d\mu$ where f is \mathcal{X} -measurable and \mathcal{X} -integrable. Under certain conditions, this specific signed measure on \mathcal{X} represents all signed measures (i.e. there are no other signed measures on \mathcal{X} that cannot be represented as the integral of the \mathcal{X} -measurable function f). This is established formally in the Radon-Nikodym Theorem which states that under certain conditions any signed measure on \mathcal{X} can be represented by the integral of a measurable function. The Radon-Nikodym Theorem will be used in the Riesz Representation Theorem in the next chapter.

Theorem 434 (Radon-Nikodym) *Let (X, \mathcal{X}, μ) be a measure space, μ be a σ -finite measure, ν be a finite signed measure on \mathcal{X} and $\nu \ll \mu$. Then there exists a \mathcal{X} -integrable function f on X such that $\nu(E) = \int_E f d\mu$ for any $E \in \mathcal{X}$. Moreover f is unique in the sense that if g is any \mathcal{X} -measurable function with this property, then $g = f$ a.e. with respect to μ .*

¹²In general, a representation theorem provides a simple way to characterize (or represent) a set of elements using certain properties that actually extends to the entire collection of elements under given assumptions.

Proof. (Sketch) By Theorem 429, a finite signed measure ν can be decomposed into ν^+ and ν^- where ν^-, ν^+ are both measures and by (ii) of Lemma 432 they are both absolutely continuous with respect to μ (if ν is). Hence it suffices to prove the theorem under the assumption that ν is a (non-negative) measure. Also, since μ is σ -finite, then X can be decomposed into countably many disjoint sets $\{E_i\}$ for which $\mu(E_i) < \infty$. Hence it suffices to prove the theorem with μ finite. In summary, we take μ, ν both finite measures with $\nu \ll \mu$.

Let G be the set of all non-negative \mathcal{X} -measurable, integrable functions g satisfying

$$\int_E g d\mu \leq \nu(E), \quad \forall E \in \mathcal{X}. \quad (5.9)$$

Among these functions g , we want to find a function f which satisfies (5.9) with equality. Since $\int_X g d\mu \leq \nu(X)$, $\forall g \in G$ (because ν is finite), the set of real numbers $\{\int g d\mu, g \in G\}$ is bounded (by $\nu(X)$) and hence its supremum exists. Let $\alpha = \sup_{g \in G} \int g d\mu$. f is constructed (using Levi's Theorem 407) as a limit function of a sequence $\langle f_n \rangle$ that attains this supremum (i.e. $\alpha = \int f d\mu$).¹³

Because $f \in G$, we know that $\int_E f d\mu \leq \nu(E)$, $\forall E \in \mathcal{X}$. We claim that $\int_E f d\mu = \nu(E)$, $\forall E \in \mathcal{X}$. If this were not true, then there would exist a set E such that $\int_E f d\mu < \nu(E)$. Then by Lemma 433, we could construct a function $g_0 = f + \varepsilon \chi_{E_0}$ belonging to G for which $\int g_0 d\mu > \alpha$. But this would violate the fact that α is the supremum. ■

The assumption in the Radon-Nikodym theorem that μ is σ -finite is important as the next exercise shows.

Exercise 5.3.4 Let $X = \mathbb{R}$ and let \mathcal{X} be a collection of all subsets of \mathbb{R} that are countable or that have countable complement. Define $\mu(E) = \begin{cases} \# \text{ of elements of } E & \text{if } E \text{ is finite} \\ \infty, & \text{otherwise} \end{cases}$ and $\nu(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } X \setminus E \text{ is countable} \end{cases}$. (i) Show that μ, ν are measures on \mathcal{X} and that $\nu \ll \mu$. (ii) Show that μ is not σ -finite. (iii) Show that the Radon-Nikodym theorem doesn't hold.

¹³In particular, by the supremum property, there exists a sequence $\langle g_n \rangle$ from G such that $\lim_{n \rightarrow \infty} \int g_n d\mu = \alpha$. Define a sequence $\langle f_n \rangle$ by $f_n = \max\{g_1, \dots, g_n\}$, $f_n \in G$. Since $\langle f_n \rangle$ is a non-decreasing sequence of integrable functions with $\int f_n d\mu \leq \alpha$, then by Levi's theorem there exists an integrable function $f = \lim f_n$ a.e. with $\int_E f d\mu = \lim_{n \rightarrow \infty} \int_E f_n d\mu \leq \nu(E)$ (because $f_n \in G$) and hence $f \in G$ and $\int f d\mu \leq \alpha$. On the other hand, because $g_n \leq f_n$ we have $\int_E f d\mu = \lim_{n \rightarrow \infty} \int_E f_n d\mu \geq \lim_{n \rightarrow \infty} \int g_n d\mu = \alpha$. Combining these two inequalities gives $\int f d\mu = \alpha$.

5.4 Examples Using Measure Theory

5.4.1 Probability Spaces

Definition 435 If $\mu(X) = 1$, then μ is a **probability measure** and (X, \mathcal{X}, μ) is called a **probability space**. In this case, X is called the **sample space**, any measurable set $A \in \mathcal{X}$ is called an **event**, and $\mu(A)$ is called the **probability of the event** A . For a probability space, we say **almost surely (a.s.)** interchangeably with **almost everywhere (a.e.)**.

We next illustrate measure spaces through some basic properties of probability.

Definition 436 Let (X, \mathcal{X}, P) be a probability space. Let Λ be an arbitrary index set and let $A_i, i \in \Lambda$ be events in \mathcal{X} . The A_i are **independent** if and only if for all finite collections $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$ we have

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}).$$

The next definition makes clear that a random variable is nothing other than a measurable function.

Definition 437 A **random variable** Y on a probability space (X, \mathcal{X}, P) is a Borel measurable function from X to \mathbb{R} (i.e. $Y : X \times \mathcal{X} \rightarrow \mathbb{R} \times \mathcal{B}(\mathbb{R})$). If Y is a random variable on (X, \mathcal{X}, P) , the **probability measure induced by** Y is the probability measure P_Y on $\mathcal{B}(\mathbb{R})$ given by $P_Y(B) = \{x \in X : Y(x) \in B\}$, $B \in \mathcal{B}(\mathbb{R})$.

The numbers $P_Y(B)$, $B \in \mathcal{B}(\mathbb{R})$, completely characterize the random variable Y in the sense that they provide the probabilities of all events involving Y . This information can be captured by a single function from \mathbb{R} to \mathbb{R} as the next definition suggests.

Definition 438 The **distribution function** of a random variable Y is the function $F : \mathbb{R} \rightarrow \mathbb{R}$ given by $F(y) = P\{x \in X : Y(x) \in B\}$.

Definition 439 If Y is a random variable on (X, \mathcal{X}, P) , the **expectation of** Y is defined by $E[Y] = \int_X Y dP$ provided the Lebesgue integral exists.

The next result gives a good illustration of simple functions, monotone convergence theorem.

Theorem 440 Let Y be a random variable on (X, \mathcal{X}, P) with distribution function F . Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function. If $Z = g \circ Y$, then $E[Z] = \int_{\mathbb{R}} g(y) dF(y)$ ($= \int_{\mathbb{R}} g dP_Y$)

Exercise 5.4.1 Prove Theorem 440 (Theorem 5.10.2 p. 223 in Ash)

One of the most remarkable results in probability is Kolmogorov's strong law of large numbers.

Theorem 441 (Strong Law of Large Numbers) If Y_1, Y_2, \dots are independent and identically distributed random variables and $E[|Y_1|] < \infty$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n Y_i = E[Y_1] \text{ a.s.}$$

5.4.2 L_1

Let us denote the collection of \mathcal{L} -integrable functions f defined on $X \subset \mathbb{R}$ by $L_1(X)$. For instance, X can be all of \mathbb{R} in which case $L_1(\mathbb{R})$ is any measurable subset of \mathbb{R} . Hence $L_1(X)$ is the collection of all \mathcal{L} -measurable functions f defined on X for which $\int_X |f| < \infty$. It is straightforward to see that $L_1(X)$ is a vector space.

Exercise 5.4.2 Show that $L_1(X)$ is a vector space. Hint: Use Theorem 403.

Can $L_1(X)$ be equipped with a norm? Let us define a function $\|\cdot\|_1 = L_1(X) \rightarrow \mathbb{R}$ given by $\|f\|_1 = \int_X |f|$. Does this function satisfy the properties of a norm given in Definition 206?

Exercise 5.4.3 Show that $\|\cdot\|_1$ satisfies properties (i) $\|f\|_1 \geq 0, \forall f \in L_1(X)$, (iii) $\|\alpha f\|_1 = |\alpha| \|f\|_1, \forall \alpha \in \mathbb{R}, f \in L_1(X)$, (iv) $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1, \forall f, g \in L_1(X)$ of the definition of a norm and the part of (ii) that $f = 0 \Rightarrow \|f\|_1 = 0$.

The next example makes it clear that the converse of part (ii) is not true.

Example 442 If f is the Dirichlet function of Example 360, then $\|f\|_1 = 0$ but $f \neq 0$ everywhere.

To overcome this problem, we will define a relation “ \sim ” on the set of all integrable functions. Let $f, g \in L_1$, define $f \sim g$ iff $f = g$ a.e. This relation is an equivalence and hence by Theorem 31 L_1 can be partitioned into disjoint classes \tilde{f} of equivalent functions (i.e. functions that are equal a.e.). Figure 5.4.1???

Exercise 5.4.4 Prove $f \sim g$ iff $f = g$ a.e is an equivalence relation using Definition 26.

By Theorem 403, for any two functions from the same equivalence class, the norm $\|f\|_1 = \|g\|_1 \equiv \|\tilde{f}\|_1$. Then the space L_1 consisting of equivalence classes with the $\|\cdot\|_1$ norm is a normed vector space.

To keep notation and terminology simple in what follows, we will refer to the elements of L_1 as functions rather than equivalence classes of functions. But you should keep in mind that when we refer to a function f we are actually referring to all functions that are equal a.e. to f .

The most important question we must ask of our new normed vector space is “Is it complete?” The next theorem provides the answer.

Theorem 443 $(L_1, \|\cdot\|_1)$ is a complete normed vector space (i.e. a Banach space).

Before proving completeness of L_1 , we note that one strategy used in previous sections is to: first, take a Cauchy sequence $\langle f_n \rangle$ in a given function space and note that for a given $x \in X$, $\langle f_n(x) \rangle$ is Cauchy in \mathbb{R} and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ exists for each x since \mathbb{R} is complete; second, prove that $f_n \rightarrow f$ with respect to the norm of the normed function space. Unfortunately, this procedure cannot be used in L_1 since for a Cauchy sequence $\langle f_n \rangle$ in L_1 a pointwise limit of $\langle f_n(x) \rangle$ may not exist for any point x as the following example shows.

Example 444 Let a sequence $\langle f_n \rangle$ of functions on $[0, 1]$ be given by

$$\begin{aligned} f_1 &= \chi_{[0, \frac{1}{2}]}, f_2 = \chi_{[\frac{1}{2}, 1]}, \\ f_3 &= \chi_{[0, \frac{1}{4}]}, f_4 = \chi_{[\frac{1}{4}, \frac{2}{4}]}, f_5 = \chi_{[\frac{2}{4}, \frac{3}{4}]}, f_6 = \chi_{[\frac{3}{4}, 1]}, \\ f_7 &= \chi_{[0, \frac{1}{8}]}, f_8 = \chi_{[\frac{1}{8}, \frac{2}{8}]}, \dots, f_{13} = \chi_{[\frac{6}{8}, \frac{7}{8}]}, f_{14} = \chi_{[\frac{7}{8}, 1]}, \\ f_{15} &= \chi_{[0, \frac{1}{16}]}, \dots \end{aligned}$$

See Figure 5.4.2. This sequence is Cauchy in $L_1([0, 1])$ but there is no point $x \in [0, 1]$ for which $\lim_{n \rightarrow \infty} f_n(x)$ exists. In other words, $\langle f_n \rangle$ doesn't converge pointwise at any point $x \in [0, 1]$.

Proof. (Sketch) Let $\langle f_n \rangle$ be a Cauchy sequence in L_1 . In order to find a function to which the sequence converges, in light of example 444, we need to take a more sophisticated approach. The fact that $\langle f_n \rangle$ is Cauchy means we can choose a subsequence $\langle f_{n_k} \rangle$ such that the two consecutive terms are so close to each other (i.e. $\|f_{n_{k+1}} - f_{n_k}\|_1 < \frac{1}{2^n}$) that their infinite sum (i.e. $\sum_{k=1}^{\infty} \int |f_{n_{k+1}} - f_{n_k}| dm$ converges a.e. on X (i.e. the sum is finite). Then by the Corollary of Levi's Theorem 408, the infinite sum $\sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$ also converges a.e. and because $f_{n_{k+1}} - f_{n_k} \leq |f_{n_{k+1}} - f_{n_k}|, \forall k$, the infinite sum $\sum_{k=1}^{\infty} f_{n_{k+1}} - f_{n_k}$ converges a.e. as well. But the sum of the differences of two consecutive terms in the subsequence itself

$$f_{n_1} + (f_{n_2} - f_{n_1}) + (f_{n_3} - f_{n_2}) + \dots + (f_{n_k} - f_{n_{k-1}}) = f_{n_k}.$$

Thus the subsequence $\langle f_{n_k} \rangle$ converges a.e. on X .

Let f be the function $\langle f_{n_k} \rangle$ which converges a.e. on X . We need to show that $\langle f_{n_k} \rangle \rightarrow f$ with respect to $\|\cdot\|_1$ and that $f \in L_1(X)$. To prove the former we can use Fatou's Lemma 393 (since $\langle f_{n_k} \rangle \rightarrow f$ a.e.) and to prove the latter we can use the estimate $\|f\|_1 \leq \|f - f_{n_k}\|_1 + \|f_{n_k}\|_1 \leq \infty$. The first term in this inequality is bounded by Fatou's Lemma and the second is bounded since $f_{n_k} \in L_1, \forall k$.

Then we have a Cauchy sequence $\langle f_n \rangle$ in $L_1(X)$ whose subsequence $\langle f_{n_k} \rangle \rightarrow f$ in $L_1(X)$. Then by Lemma 173, the whole sequence $\langle f_n \rangle \rightarrow f$ in $L_1(X)$. ■

Approximation in L_1

The next theorem establishes that the simple and continuous functions are dense in $L_1(X)$.

Theorem 445 *Let f be an \mathcal{L} -integrable function on \mathbb{R} and let $\varepsilon > 0$. Then (i) there is an integrable simple function φ such that $\int |f - \varphi| < \varepsilon$ and (ii) there is a continuous function g such that g vanishes (i.e. $g = 0$) outside some bounded interval and such that $\int |f - g| < \frac{\varepsilon}{2}$.*

Proof. (of i) Without loss of generality, we may assume that $f \geq 0$ (otherwise $f = f^+ - f^-$ where f^+ and f^- are non-negative). If f is \mathcal{L} -integrable,

then using the supremum property in Definition 388 for $\varepsilon > 0$, there exists a bounded \mathcal{L} -measurable function h that vanishes outside a set E of finite measure (i.e. $m(E) < \infty$) such that $h \leq f$ and $(\int f) - \varepsilon < \int h \Leftrightarrow \int (f - h) < \varepsilon$. Then by Theorem 367 there is a non-decreasing sequence (since h is bounded) of simple functions $\langle h_n \rangle$ converging uniformly to h . Then for $\frac{\varepsilon}{2m(E)} > 0$, $\exists N$ such that $|h_n(x) - h(x)| < \frac{\varepsilon}{2m(E)}$ for all $x \in E$. Hence

$$\int_E |f - h_N| \leq \int_E |f - h| + \int_E |h_N - h| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2m(E)}m(E) = \varepsilon.$$

(of ii) Given: g , \exists continuous function h s.t. $g(x) = R(x)$, except on a set $\leq \frac{\varepsilon}{3}$. f is integrable

$$\int_X f dx = \inf_{\psi \geq f} \int_X \psi dx \quad (5.10)$$

, $\forall \varepsilon > 0$, $\exists \int |f - \psi| dx < \varepsilon$. Referring to equation (5.10), ψ is a simple function i.e. $\psi(x) = \sum_{i=1}^n a_i \chi_{E_i}(x)$, $\mu(E_i) < \infty$, $\forall i$ if μ is Lebesgue measure. $\mu(M) < \infty$, then for $\varepsilon > 0$, $\exists F_M$ closed and G_M open such that $F_M \subset M \subset G_M$, $\mu(G_M) - \mu(F_M) < \varepsilon$. Let's define

$$\varphi_\varepsilon(x) = \frac{\rho(x, \mathbb{R} \setminus G_M)}{\rho(x, \mathbb{R} \setminus G_M) + \rho(x, F_M)}$$

If

$$\varphi_\varepsilon(x) = 0 \text{ if } x \in \mathbb{R} \setminus G_M$$

$\varphi_\varepsilon(x) = 1$ if $x \in F_M$ $\varphi_\varepsilon(x)$ is continuous because $\rho(x, \mathbb{R} \setminus G_M)$ and $\rho(x, F_M)$ are continuous and $\rho(x, \mathbb{R} \setminus G_M) + \rho(x, F_M) \neq 0$. Function

$$\chi_M - \rho_\varepsilon = \chi_{\mathbb{R} \setminus G_M \setminus F_M}$$

$\chi_M - \rho_\varepsilon = 0$ on $\mathbb{R} \setminus (G_M \setminus F_M)$ Hence

$$\int |\chi_M(x) - \rho_\varepsilon(x)| d\mu < \varepsilon$$

Thus χ_M is approximated by a continuous function ρ_ε . ■

Separability of $L_1(X)$

In the next chapter we will show that if X is compact, then $L_1(X)$ is separable with the countable dense set being the set of all polynomials with rational coefficients. But if X is an arbitrary \mathcal{L} -measurable set including X with $m(X) = \infty$ (i.e. $X = \mathbb{R}$) we need to find a different countably dense set. We will show that the set M of all finite linear combinations of the form

$$\sum_{i=1}^n c_i \chi_{I_i} \quad (5.11)$$

where the numbers c_i , $i = 1, \dots, n$ are rational and I_i are all intervals (open, closed, and half-open) with rational endpoints is a countably dense set in $L_1(X)$.

Countability of M is obvious. We need to show that M is dense in $L_1(X)$. By Theorem 445 we know that the set of all integrable simple functions is dense in $L_1(X)$. But every such function can be approximated arbitrarily closely by a function of the same type taking only rational values. Thus given $f \in L_1(X)$ and $\varepsilon > 0$ there is an integrable simple function $\varphi = \sum_{i=1}^n y_i \chi_{E_i}$ where y_i are rational coefficients, E_i are mutually disjoint \mathcal{L} -measurable sets, and $\cup_{i=1}^n E_i = X$ such that $\int_X |f - \varphi| dm < \varepsilon$. If the function φ were of the type (5.11) we would be done. Unfortunately it is not because it requires E_i to be intervals (recall that the collection of all intervals with rational endpoints is countable whereas the collection of \mathcal{L} -measurable subsets of X may not be countable). Hence we need to show that every simple integrable function φ can be approximated by functions of the form (5.11). Here we use the fact that if a set E is \mathcal{L} -measurable then it can be approximated by an interval (i.e. given $\varepsilon > 0$, there is an interval I such that¹⁴

$$m((E \setminus I) \cup (I \setminus E)) < \varepsilon. \quad (5.12)$$

Now using (5.12) for sets $\{E_i\}_{i=1}^n$ we can construct $\{I_i\}_{i=1}^n$ such that $m((E \setminus I_i) \cup (I_i \setminus E)) < \varepsilon$ for $i = 1, \dots, n$. Let $\widehat{I}_i = I_i \setminus \cup_{j < i} I_j$, $i = 1, \dots, n$. Then \widehat{I}_i are mutually disjoint. Define a function

$$\psi(x) = \begin{cases} y_i & \text{if } x \in \widehat{I}_i \\ 0 & \text{if } x \in X \setminus \cup_{i=1}^n \widehat{I}_i \end{cases}.$$

¹⁴We proved a similar result in Theorem 347 where a measurable set E is approximated by open and closed sets.

The function φ and ψ differ from each other on a set B with sufficiently small measure, namely $m(B) = m(x \in X : \varphi(x) \neq \psi(x)) < n\varepsilon$. Hence

$$\begin{aligned} \|\varphi - \psi\|_1 &= \int_X |\varphi(x) - \psi(x)| dm = \int_B |\varphi(x) - \psi(x)| \\ &\leq \sup_n |y_n| m(B) < (n \sup_n |y_n|) \varepsilon \end{aligned}$$

can be made arbitrarily small by choosing ε sufficiently small. Thus ψ approximates φ and ψ is of the form (5.11). Thus we have the following theorem.

Theorem 446 $L_1(X)$ is separable.

Proof. The countable dense set in $L_1(X)$ is the set given by (5.11). ■

5.5 Appendix - Proofs in Chapter 5

Proof of Theorem 329. Take a closed finite interval $[a, b]$. Since $[a, b] \subset (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$, then $m^*([a, b]) \leq l(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}) = b - a + \varepsilon, \forall \varepsilon > 0$ so that $m^*([a, b]) \leq b - a$. Next we will show that $m^*[a, b] \geq b - a$. But this is equivalent to showing that if $\{I_n\}_{n \in \mathbb{N}}$ is an open covering of $[a, b]$, then $\sum_{n \in \mathbb{N}} l(I_n) \geq b - a$. By the Heine-Borel Theorem 194 there is a finite subcollection that also covers $[a, b]$. Since the sum of the lengths of the finite subcollection can be no greater than the lengths of the original collection, it suffices to show $\sum_{n=1}^N l(I_n) \geq b - a$ for N finite. It is possible to construct a finite sequence of open intervals $\langle (a_k, b_k) \rangle_{k=1}^K$ with $a_k < b_{k-1} < b_k$ such that $a \in (a_1, b_1)$ and $b \in (a_K, b_K)$.¹⁵ Thus

$$\begin{aligned} \sum_{n \in \mathbb{N}} l(I_n) &\geq \sum_{k=1}^K l(a_k, b_k) \\ &= b_K - (a_K - b_{K-1}) - (a_{K-1} - b_{K-2}) - \dots - (a_2 - b_1) - a_1 \\ &\geq b_K - a_1 \\ &\geq b - a. \end{aligned}$$

or $m^*([a, b]) \geq b - a$. Thus $m^*([a, b]) = b - a$.

¹⁵Since $a \in \cup_{n=1}^N I_n$, $\exists (a_1, b_1)$ such that $a \in (a_1, b_1)$. If $b_1 \leq b$, then since $b_1 \notin (a_1, b_1)$, $\exists (a_2, b_2)$ such that $b_1 \in (a_2, b_2)$. Continue by induction.

To complete the proof we simply need to recognize that if I is *any* finite interval, then for a given $\varepsilon > 0$, there is a closed interval $[a, b] \subset I$ such that $l(I) - \varepsilon < l([a, b])$. Hence,

$$l(I) - \varepsilon < l([a, b]) = m^*([a, b]) \leq m^*(I) \leq m^*(\bar{I}) = l(\bar{I}) = l(I)$$

where the first equality follows from the first part of this theorem, the first weak inequality follows from monotonicity in Theorem 328, the second weak inequality follows from the definition of closure, and the next two equalities follow from the definition of length. Since $l(I) - \varepsilon < m^*(I) \leq l(I)$ and $\varepsilon > 0$ is arbitrary, taking $\varepsilon \rightarrow 0$ gives $m^*(I) = l(I)$. If I is an infinite interval, $m^*(I) = \infty$. ■

Proof of Theorem 331. If $m^*(A_n) = \infty$ for any n , then the inequality holds trivially. Assume $m^*A_n < \infty, \forall n$. Then given $\varepsilon > 0$, boundedness implies that for each n , we can choose intervals $\{I_k^n\}_{k \in \mathbb{N}}$ such that $A_n \subset \cup_{k \in \mathbb{N}} I_k^n$ (i.e. the intervals cover A_n) and $\sum_{k \in \mathbb{N}} l(I_k^n) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$.¹⁶ But the collection $\{I_k^n\}_{n,k} = (\cup_{n \in \mathbb{N}} \{I_k^n\}_{k \in \mathbb{N}})$ is countable, being the union of a countable number of countable collections and covers $\cup_{n \in \mathbb{N}} A_n$ (i.e. $\cup_{n \in \mathbb{N}} A_n \subset \cup_{n \in \mathbb{N}} \cup_{k \in \mathbb{N}} I_k^n$). Hence

$$m^*(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} l(I_k^n) \leq \sum_{n \in \mathbb{N}} \left(m^*(A_n) + \frac{\varepsilon}{2^n} \right) = \sum_{n \in \mathbb{N}} m^*(A_n) + \varepsilon.$$

Subadditivity follows since $\varepsilon \geq 0$ was arbitrary and we can let $\varepsilon \rightarrow 0$. ■

Proof of Theorem 341. Corollary 338 already established that \mathcal{L} is an algebra. Hence it is sufficient to prove that if a set $E = \cup_{n \in \mathbb{N}} E_n$ where each E_n is \mathcal{L} -measurable, then E is \mathcal{L} -measurable. By Theorem 84, we may assume without loss of generality that the E_n are mutually disjoint sets.

Let A be any set and $F_N = \cup_{n=1}^N E_n$. Since \mathcal{L} is an algebra and E_1, \dots, E_N are in \mathcal{L} , the sets F_N are \mathcal{L} -measurable. For any set A , we have

$$\begin{aligned} m^*(A) &= m^*(A \cap F_N) + m^*(A \cap F_N^c) & (5.13) \\ &\geq m^*(A \cap F_N) + m^*(A \cap E^c) \\ &= \sum_{n=1}^N m^*(A \cap E_n) + m^*(A \cap E^c) \end{aligned}$$

¹⁶Existence of the countable collection follows from Theorem 108 and the inequality holds by the property of the infimum that $\underline{x} = \inf A \Rightarrow \forall \varepsilon > 0, \exists x \in A$ such that $x < \underline{x} + \varepsilon$.

where the first equality follows by Definition 334, the inequality follows since $F_N^c \supset E^{c17}$, and the last equality follows by Lemma 339. Since the left hand side of (5.13) is independent of N , letting $N \rightarrow \infty$ we have

$$\begin{aligned} m^*(A) &\geq \sum_{n=1}^{\infty} m^*(A \cap E_n) + m^*(A \cap E^c) \\ &\geq m^*(A \cap E) + m^*(A \cap E^c) \end{aligned} \quad (5.14)$$

Here the second inequality follows from Theorem 331. But (5.14) is simply the sufficient condition for E to be \mathcal{L} -measurable. ■

Proof of Theorem 345. Let A be any set, $A_1 = A \cap (a, \infty)$, and $A_2 = A \cap (-\infty, a]$. According to (5.1), it is sufficient to show $m^*(A) \geq m^*(A_1) + m^*(A_2)$. If $m^*A = \infty$, the assertion is trivially true. If $m^*(A) < \infty$, then for each $\varepsilon > 0$ there is countable collection $\{I_n\}$ of open intervals which cover A and for which $\sum_{n=1}^{\infty} l(I_n) \leq m^*(A) + \varepsilon$ by the infimum property in Definition 327. Let $I'_n = I_n \cap (a, \infty)$ and $I''_n = I_n \cap (-\infty, a]$. Then $I'_n \cup I''_n = I_n \cap \mathbb{R} = I_n$ and $I'_n \cap I''_n = \emptyset$. Therefore, $l(I_n) = l(I'_n) + l(I''_n) = m^*(I'_n) + m^*(I''_n)$. Since $A_1 \subset (\cup_{n=1}^{\infty} I'_n)$, then $m^*(A_1) = m^*(\cup_{n=1}^{\infty} I'_n) \leq \sum_{n=1}^{\infty} m^*(I'_n)$. Similarly, since $A_2 \subset (\cup_{n=1}^{\infty} I''_n)$, then $m^*(A_2) = m^*(\cup_{n=1}^{\infty} I''_n) \leq \sum_{n=1}^{\infty} m^*(I''_n)$. Thus,

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum_{n=1}^{\infty} [m^*(I'_n) + m^*(I''_n)] \\ &\leq \sum_{n=1}^{\infty} l(I_n) \leq m^*(A) + \varepsilon. \end{aligned}$$

But since $\varepsilon > 0$ was arbitrary, the result follows. ■

Proof of Measurable Selection Theorem 371. By induction, we will define a sequence of measurable functions $f_n : X \rightarrow Y$ such that

- (i) $d(f_n(z), \Gamma(z)) < \frac{1}{2^n}$ and
- (ii) $d(f_{n+1}(z), f_n(z)) \leq \frac{1}{2^{n-1}}$ on X for all n .

Then we are done, since from (ii) it follows that $\langle f_n \rangle$ is Cauchy and due to completeness of Y there exists a function $f : X \rightarrow Y$ such that $f_n(z) \rightarrow f(z)$ on X and by Corollary 358 the pointwise limit of a sequence of measurable functions is measurable. Condition (i) guarantees that $f(z) \in \Gamma(z), \forall z \in X$ where f is a measurable selection (here we use the fact that $\Gamma(z)$ is closed and $d(f(z), \Gamma(z)) = 0$ implies $f(z) \in \Gamma(z)$).

¹⁷Recall by DeMorgan's Law that $F_N^c = [\cup_{n=1}^N E_n]^c = \cap_{n=1}^N E_n^c$.

Now we construct a sequence $\langle f_n \rangle$ of measurable functions satisfying (i) and (ii). Let $\{y_n, n \in \mathbb{N}\}$ be a dense set in Y (since Y is separable such a countable set exists). Define $f_0(z) = y_p$ where p is the smallest integer such that $\Gamma(z) \cap B_1(y_p) \neq \emptyset$ ($f_0(z)$ is well defined because $\{y_n, n \in \mathbb{N}\}$ is dense in Y). Since Γ is measurable then

$$f_0^{-1}(y_p) = [\Gamma^{-1}(B_1(y_p))] \setminus [\cup_{m < p} \Gamma^{-1}(B_1(y_m))] \in \mathcal{L}.$$

Let V be open in Y . Then $f_0^{-1}(V)$ is in at most a countable union of such $f_0^{-1}(y_p)$. Hence $f_0^{-1}(V)$ is measurable so that f_0 is measurable. Suppose we already have f_k measurable. Then $z \in f_k^{-1}(y_i) \equiv D_i$ implies $f_k(z) = y_i$ and $d(f_k(z), \Gamma(z)) < \frac{1}{2^k}$ (i.e. $\Gamma(z) \cap B_{2^{-k}}(y_i) \neq \emptyset$). Therefore we can define $f_{k+1}(z) = y_p$ for $z \in D_i$ where p is the smallest integer such that $\Gamma(z) \cap B_{2^{-k}}(y_i) \cap B_{2^{-k-1}}(y_p) \neq \emptyset$. Thus f_{k+1} is defined on $X = \cup_{i \geq 1} D_i$, it is measurable, and we have $d(f_{k+1}(z), \Gamma(z)) < \frac{1}{2^{k+1}}$ and $d(f_{k+1}(z), f_k(z)) \leq \frac{1}{2^k} + \frac{1}{2^{k+1}} \leq \frac{1}{2^{k-1}}$ on X . ■

Proof of Theorem 382. (\Leftarrow) If f is \mathcal{L} -measurable and bounded by M , then we can construct sets

$$E_k = \left\{ x \in E : \frac{kM}{n} \geq f(x) > \frac{(k-1)M}{n} \right\}, \quad -n \leq k \leq n$$

which are measurable, disjoint, and have union E . Thus $\sum_{k=-n}^n mE_k = mE$. Define simple functions $\psi_n(x) = \frac{M}{n} \sum_{k=-n}^n k \chi_{E_k}(x)$ and $\varphi_n(x) = \frac{M}{n} \sum_{k=-n}^n (k-1) \chi_{E_k}(x)$. Then $\psi_n(x) \geq f(x) \geq \varphi_n(x)$. Thus

$$L^u \int_E f(x) dx = \inf_{\psi \geq f} \int_E \psi(x) dx \leq \int_E \psi_n(x) dx = \frac{M}{n} \sum_{k=-n}^n kmE_k \quad (5.15)$$

and

$$L^l \int_E f(x) dx = \sup_{\varphi \leq f} \int_E \varphi(x) dx \geq \int_E \varphi_n(x) dx = \frac{M}{n} \sum_{k=-n}^n (k-1)mE_k. \quad (5.16)$$

Then (5.15)-(5.16) implies

$$0 \leq L^u \int_E f(x) dx - L^l \int_E f(x) dx \leq \frac{M}{n} \sum_{k=-n}^n mE_k = \frac{M}{n} mE.$$

Since $mE < \infty$ by assumption, $\lim_{n \rightarrow \infty} \frac{M}{n} mE = 0$. ■ ■

Proof of Bounded Convergence Theorem 386. Since f is the limit of \mathcal{L} -measurable functions f_n it is \mathcal{L} -measurable by Theorem 364 and hence integrable. By Theorem 366 we know that given $\varepsilon > 0$, $\exists N$ and an \mathcal{L} -measurable set $A \subset E$ with $mA < \frac{\varepsilon}{4M}$ such that for $n \geq N$ and $x \in E \setminus A$ we have $|f_n(x) - f(x)| < \frac{\varepsilon}{2mE}$. Furthermore, since $|f_n(x)| \leq M$, $\forall n \in \mathbb{N}$ and $\forall x \in E$, then $|f(x)| \leq M$, $\forall x \in E$ and $|f_n(x) - f(x)| \leq 2M$, $\forall x \in A$. Therefore,

$$\begin{aligned} \left| \int_E f_n - \int_E f \right| &= \left| \int_E (f_n - f) \right| \\ &\leq \int_E |f_n - f| \\ &= \int_{E \setminus A} |f_n - f| + \int_A |f_n - f| \\ &< \frac{\varepsilon}{2mE} m(E \setminus A) + 2MmA < \varepsilon, \forall n \geq N. \end{aligned}$$

where the first inequality follows by monotonicity (i.e. (iii) of Theorem 385) and the second equality follows from (v) of Theorem 385). Hence $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$. ■ ■

Proof of Fatou's Lemma 393. WLOG we may assume the convergence is everywhere since integrals over sets of measure zero are zero. Let h be a bounded \mathcal{L} -measurable function such that $h \leq f$ and vanishes outside a set $H = \{x \in E : h(x) \neq 0\}$ of finite measure (i.e. $mH < \infty$). Define a sequence of functions $h_n(x) = \min\{h(x), f_n(x)\}$. Then h_n is bounded (by the bound for h) and vanishes outside H . Moreover, $\lim_{n \rightarrow \infty} h_n(x) = \lim_{n \rightarrow \infty} \min\{h(x), f_n(x)\} = \min\{h(x), f(x)\} = h(x)$ on H . Since $\langle h_n \rangle$ is a uniformly bounded sequence of \mathcal{L} -measurable functions such that $h_n \rightarrow h$, then $\lim_{n \rightarrow \infty} \int_H h_n = \int_H h$ by the Bounded Convergence Theorem 386. Since h vanishes outside H , then $\int_E h = \int_H h$.

While $f_n \rightarrow f$ a.e., we do not have that the sequence $\langle \int_E f_n(x) \rangle$ is convergent. However,

$$\int_E h = \lim_{n \rightarrow \infty} \int_H h_n = \underline{\lim}_{n \rightarrow \infty} \int_H h_n \leq \underline{\lim}_{n \rightarrow \infty} \int_E f_n$$

where the second equality follows from the fact that $\liminf = \limsup$ at a limit point and the inequality follows since $h_n(x) \leq f_n(x)$ by construction.¹⁸

¹⁸We chose \liminf rather than \limsup since this gives a tighter bound.

Taking the supremum over all $h \leq f$ we have

$$\sup_{h \leq f} \int_E h = \int_E f \leq \underline{\lim}_{n \rightarrow \infty} \int_E f_n$$

where the equality uses Definition 388. ■ ■

Proof of Lebesgue Dominated Convergence Theorem 404. By Lemma 402, f_n is integrable over E . Since $\lim_{n \rightarrow \infty} f_n = f$ a.e. on E and $|f_n| \leq g$, then $|f| \leq g$ a.e. on E . Hence f is integrable over E .

Now consider a sequence $\langle h_n \rangle$ of functions defined by $h_n = f_n + g$ which is nonnegative by construction and integrable for each n . Therefore, by Fatou's Lemma 393, we have $\int_E (f + g) \leq \underline{\lim}_{n \rightarrow \infty} \int_E (f_n + g)$, which implies $\int_E f \leq \underline{\lim}_{n \rightarrow \infty} \int_E f_n$ by (ii) of Theorem 403.

Similarly, construct the sequence $\langle k_n \rangle$ of functions defined by $k_n = g - f_n$ which is again nonnegative by construction and integrable for each n . Therefore, by Fatou's Lemma 393, we have $\int_E (g - f) \leq \underline{\lim}_{n \rightarrow \infty} \int_E (g - f_n)$, which implies $\int_E f \geq \overline{\lim}_{n \rightarrow \infty} \int_E f_n$ by (ii) of Theorem 403. ■

Proof of Levi's Theorem. 407 Assume that $f_1 \geq 0$ (otherwise we would consider $\bar{f}_n = f_n - f_1$). Define $\Omega = \{x \in A : f_n(x) \rightarrow +\infty\}$. Then $\Omega = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \Omega_n^{(r)}$, where $\Omega_n^{(r)} = \{x \in A : f_n(x) > r\}$. Using the Chebyshev's inequality (Lemma ??) $m(\Omega_n^{(r)}) \leq \frac{K}{r}$. Since $\Omega_1^{(r)} \subset \Omega_2^{(r)} \subset \dots \subset \Omega_n^{(r)} \subset \dots$, this implies $m(\bigcup_{n=1}^{\infty} \Omega_n^{(r)}) \leq \frac{K}{r}$. But for any r we have $\Omega \subset \bigcup_{n=1}^{\infty} \Omega_n^{(r)}$. Then $m(\Omega) \leq \frac{K}{r}$. Since r was arbitrary, we have $m(\Omega) = 0$. Thus we have proved that $\langle f_n \rangle \rightarrow f$ a.e. on A . Let's define $A_r = \{x \in A : r - 1 \leq f(x) < r, r \in \mathbb{N}\}$ and let $\varphi(x) = r$ on A_r . If we prove that φ is integrable on A then using the Lebesgue Dominated Convergence Theorem 404 we can conclude that Levi's theorem holds. Let $B_s = \bigcup_{r=1}^s A_r$. Since on B_s , f_n and f are bounded and $\varphi(x) \leq f(x) + 1$, we have $\int_{B_s} \varphi(x) dm \leq \int_{B_s} f(x) dm + m(A) = \lim_{n \rightarrow \infty} \int_{B_s} f_n(x) dm + m(A) \leq K + m(A)$ where $\int_{B_s} \varphi(x) dm = \sum_{r=1}^s r m(A_r)$. Hence we have $\sum_{r=1}^s r m(A_r) \leq K + m(A)$ for any s . Boundedness of partial sums of a series means that the infinite series $\sum_{r=1}^{\infty} r m(A_r)$ exists and equals $\int_A \varphi(x) dm$. ■

Proof of Hahn Decomposition Theorem 427. Due to Zorn's lemma a maximal system \mathcal{E} of disjoint measurable sets E with $\nu(E) < 0$ exists. Moreover \mathcal{E} is countable (by Lemma 422). Put $B = \bigcup \{E \in \mathcal{E}\}$. Then B is measurable and negative (because all of its subsets have negative sign measure).

Let $A = X \setminus B$. We have $A \cap B = \emptyset$, $A \cup B = X$ and A is measurable. We have to show that A is positive. By contradiction assume that A is not positive. Then there exists

$$E_0 \in \mathcal{X} \text{ such that } E_0 \subset A \text{ and } \nu(E_0) < 0. \quad (5.17)$$

Let \mathcal{E}_0 denote the maximal collection of disjoint measurable sets $F \subset E_0$ for which $\nu(F) > 0$ (at least one such set F exists because otherwise E_0 would be negative and would contradict maximality of \mathcal{E}). Due to Lemma 422 the collection \mathcal{E}_0 is countable. Let $F_0 = \cup \{F \in \mathcal{E}_0\}$. We have

$$\nu(F_0) > 0 \text{ and } F_0 \subset E_0. \quad (5.18)$$

It follows that $E_0 \setminus F_0$ is negative because, by construction, it doesn't contain a positive measurable set. Then from the equality $E_0 = (E_0 \setminus F_0) \cup F_0$ we have

$$\nu(E_0) = \nu(E_0 \setminus F_0) + \nu(F_0). \quad (5.19)$$

From (5.17), (5.18), and (5.19) we have $\nu(E_0 \setminus F_0) < 0$. The set $E_0 \setminus F_0$ is then negative, with negative measure, and $(E_0 \setminus F_0) \cap B = \emptyset$ which contradicts the maximality of set B . ■

Proof of Radon-Nikodym Theorem 434. By (ii) of Lemma 432, it suffices to deal with ν which is non-negative (i.e. a measure). Also since μ is σ -finite, the whole space X can be decomposed into countably many disjoint sets $\{E_i\}$ for which $\mu(E_i) < \infty$. Hence in the proof we can assume that μ and ν are both finite measures on \mathcal{X} . Let G be the set of all non-negative, \mathcal{X} -measurable, integrable functions g for which

$$\int_E g d\mu \leq \nu(E), \quad \forall E \in \mathcal{X}.$$

Setting $E = X$ we have

$$\int_X g d\mu \equiv \int g d\mu \leq \nu(X), \quad \forall g \in G.$$

Hence the set of real numbers $\{\int g d\mu, g \in G\}$ is bounded from above by $\nu(X)$ and thus there exists a real number α such that

$$\alpha = \sup_{g \in G} \int g d\mu.$$

Then by the supremum property, there exists a sequence $\langle g_n \rangle_{n=1}^{\infty}$ from G such that

$$\lim_{n \rightarrow \infty} \int g_n d\mu = \alpha.$$

Let us set for $n \in \mathbb{N}$, $f_n = \max\{g_1, \dots, g_n\}$. Clearly $f_n \leq f_{n+1}$ for $n \in \mathbb{N}$.

Next we show that $f_n \in G$. It suffices to show that if $g_1, g_2 \in G$, then $\max\{g_1, g_2\} \in G$. Given a set $E \in \mathcal{X}$, define

$$\begin{aligned} F &= \{x \in E : \max(g_1, g_2)(x) = g_1(x)\} \\ G &= \{x \in E : \max(g_1, g_2)(x) = g_2(x) \neq g_1(x)\}. \end{aligned}$$

Then because F and G are disjoint we have $\int_E \max(g_1, g_2) d\mu = \int_{F \cup G} \max(g_1, g_2) d\mu = \int_F g_1 d\mu + \int_G g_2 d\mu \leq \nu(F) + \nu(G) = \nu(E)$. Thus $\max(g_1, g_2) \in G$ and consequently each function f_n , $n \in \mathbb{N}$ belongs to G .

Since $\alpha = \sup_{g \in G} \int g d\mu$, then $\int f_n d\mu \leq \alpha$, for $n \in \mathbb{N}$. Let $f : X \rightarrow \mathbb{R}$ be defined as $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. This is a well defined function because for any $x \in X$, the sequence $\langle f_n(x) \rangle$ is non-decreasing and hence $\lim f_n(x)$ exists. Then by Levi's Theorem 407 f is integrable and $\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$.

We next show that $f \in G$. For $E \in \mathcal{X}$ and $n \in \mathbb{N}$ we have $\chi_E f_n \leq \chi_E f_{n+1}$ and $\lim_{n \rightarrow \infty} \chi_E(x) f_n(x) = \chi_E(x) f(x)$ for all $x \in \mathcal{X}$. Then

$$\int_E f d\mu = \int f \chi_E d\mu = \int \lim_{n \rightarrow \infty} f_n \chi_E d\mu = \lim_{n \rightarrow \infty} \int f_n \chi_E d\mu = \lim_{n \rightarrow \infty} \int_E f_n d\mu \leq \nu(E). \quad (5.20)$$

This shows that $f \in G$ and hence $\int f d\mu \leq \alpha$ because $f_n \geq g_n$ for $n \in \mathbb{N}$ we have $\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu \geq \lim_{n \rightarrow \infty} \int g_n d\mu = \alpha$. Combining the last

two inequalities we have $\int f d\mu = \alpha$.

We now show that $\int_E f d\mu = \nu(E)$ for all $E \in \mathcal{X}$. By contradiction, assume this equality doesn't hold. Then due to (5.20), $\int_E f d\mu < \nu(E)$ and then the set function

$$\nu_0(E) = \nu(E) - \int_E f d\mu \quad (5.21)$$

is a finite measure not identically equal to zero. Since $\nu_0 \ll \mu$ (because $\nu \ll \mu$ and $\int f d\mu \ll \mu$) using Lemma 433 there exists $\varepsilon > 0$ and $E_0 \in \mathcal{X}$ such that $\mu(E_0) > 0$ and

$$\varepsilon \mu(E_0 \cap F) \leq \nu_0(E_0 \cap F), \forall F \in \mathcal{X}. \quad (5.22)$$

Set $g_0 = f + \varepsilon \chi_{E_0}$. We have $\int g_0 d\mu = \int f d\mu + \varepsilon \mu(E_0) > \int f d\mu = \alpha$. If we show that $g_0 \in G$ this would contradict the fact that $\alpha = \sup_{g \in G} \int g d\mu$. For any $F \in \mathcal{X}$ and using (5.21) and (5.22) we have $\int_F g_0 d\mu = \int_F (f + \varepsilon \chi_{E_0}) d\mu = \int_F f d\mu + \varepsilon \mu(E_0 \cap F) \leq \int_F f d\mu + \nu_0(E_0 \cap F) = \int_F f d\mu + \nu(E_0 \cap F) - \int_{E_0 \cap F} f d\mu = \int_{F \setminus (E_0 \cap F)} f d\mu + \nu(E_0 \cap F) \leq \nu(F \setminus (E_0 \cap F)) + \nu(E_0 \cap F) = \nu(F)$. Hence $g_0 \in G$, which leads to the contradiction. The uniqueness of f (except on a set of measure zero) follows from Exercise 5.2.4. ■

Proof of Theorem 443. Let $\langle f_n \rangle$ be a Cauchy sequence in L_1 so that $\|f_m - f_n\|_1 \rightarrow 0$ as $m, n \rightarrow \infty$. Then we can find a sequence of indices $\langle n_k \rangle$ with $n_1 < n_2 < \dots < n_k < \dots$ such that

$$\|f_{n_k} - f_{n_{k-1}}\|_1 = \int_X |f_{n_k} - f_{n_{k-1}}| dm < \frac{1}{2^k}, k = 1, 2, \dots \quad (5.23)$$

Define a sequence $\langle g_k \rangle$ by $g_k = f_{n_k} - f_{n_{k-1}}$ for $k = 2, 3, \dots$ with $g_1 = f_{n_1}$. Then (5.23) is simply $\int_X |g_k| dm < \frac{1}{2^k}$ and taking the infinite sum of both sides yields

$$\sum_{k=1}^{\infty} \int_X |g_k| dm \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Thus by the Corollary of Levi's Theorem 408, $\sum_{k=1}^{\infty} |g_k|$ converges a.e. on X . Since $g_k \leq |g_k|$, then $\sum_{k=1}^{\infty} g_k$ converges a.e. on X (i.e. there exists a function f such that $\sum_{k=1}^{\infty} g_k \rightarrow f$ a.e. on X). But

$$\begin{aligned} f &= \sum_{k=1}^{\infty} g_k = \lim_{J \rightarrow \infty} \sum_{k=1}^J g_k = \lim_{J \rightarrow \infty} (g_1 + g_2 + \dots + g_J) \\ &= \lim_{J \rightarrow \infty} [f_{n_1} + (f_{n_2} - f_{n_1}) + (f_{n_3} - f_{n_2}) + \dots + (f_{n_J} - f_{n_{J-1}})] = \lim_{J \rightarrow \infty} f_{n_J}. \end{aligned}$$

Hence $\langle f_{n_k} \rangle \rightarrow f$ a.e. on X .

Now we show that $\langle f_{n_k} \rangle \rightarrow f$ with respect to $\|\cdot\|_1$ and that $f \in L_1(X)$. Since $\langle f_{n_k} \rangle$ is Cauchy in $L_1(X)$ (and a subsequence of a Cauchy sequence is Cauchy), given $\varepsilon > 0$,

$$\int |f_{n_k} - f_{n_l}| dm < \varepsilon \quad (5.24)$$

for sufficiently large k and l . Hence by Fatou's Lemma 393 we can take the limit as $l \rightarrow \infty$ behind the integral in (5.24) obtaining

$$\int |f_{n_k}(x) - f(x)| dm = \|f - f_{n_k}\|_1 \leq \varepsilon.$$

Since

$$\|f\|_1 = \|f - f_{n_k} + f_{n_k}\|_1 \leq \|f - f_{n_k}\|_1 + \|f_{n_k}\|_1 \leq \varepsilon + \|f_{n_k}\|_1 < \infty$$

it follows that $f \in L_1(X)$ and $\langle f_{n_k} \rangle \rightarrow f$ in $L_1(X)$. But by Lemma 173, if a Cauchy sequence contains a subsequence converging to a limit, then the sequence itself converges to the same limit. Hence $\langle f_n \rangle \rightarrow f$ in $L_1(X)$. ■

Figures for Sections X to X

5.6 Bibliography for Chapter 5

This material is based on Royden (Chapters 3, 4, 11, 12) and Jain and Gupta (1986) *Lebesgue Measure and Integration*, New York: Wiley (Chapters 3 to 5).

Chapter 6

Function Spaces

In this chapter we will consider applications of functional analysis in economics such as dynamic programming, existence of equilibrium price functionals, and approximation of functions. In the first case, we can represent complicated sequence problems such as the optimal growth model by a simple functional equation. In particular, a representative household's lifetime utility, conditional on an initial level of capital k is denoted by the function $v(k)$ which solves

$$v(k) = \max_{k' \in [0, K]} u(f(k) - k') + \beta v(k')$$

where k' denotes next period's choice of capital which lies in some compact set $X = [0, K]$, $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is an increasing, continuous function representing the household's preferences over consumption which is just output that is not saved for next period (i.e. $f(k) - k'$) and $\beta \in (0, 1)$ represents the fact that households discount the future. We can think of the above equation as defining an operator T which maps continuous functions defined on a compact set (the $v(k')$ on the right hand side of the above equation) into continuous functions (the $v(k)$ on the left hand side). If we let $\mathcal{C}(X)$ denote the set of continuous functions defined on the compact set X , then we have $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$. In this chapter we analyse under what conditions solutions to such functional equations exist. Another simple example of such operators from mathematics are differential equations.

Let (X, d_X) and (Y, d_Y) be metric spaces. In Chapter 4 we studied functions f that took points in a metric space (X, d_X) into points in a metric space (Y, d_Y) . Now let $\mathcal{F}(X, Y)$ denote the collection of all such functions

$f : X \rightarrow Y$. Let $\mathfrak{B}(X, Y)$ be a subset of $\mathcal{F}(X, Y)$ with the property that for each pair $f, g \in \mathfrak{B}(X, Y)$, the set $\{d_Y(f(x), g(x)) : x \in X\}$ is bounded. We say $\mathfrak{B}(X, Y)$ is the collection of all bounded functions.

Definition 447 Given $\mathfrak{B}(X, Y)$, define a metric $\bar{d} : \mathfrak{B} \times \mathfrak{B} \rightarrow \mathbb{R}$ by

$$\bar{d}(f, g) = \sup\{d_Y(f(x), g(x)) : x \in X\}.$$

This metric is called the **sup (supremum) metric**. See Figure 6.1

Exercise 6.0.1 Show that \bar{d} is a metric. To do so, see Definition 125.

Example 448 The existence of the metric \bar{d} (i.e. of the supremum) is guaranteed only if the space is bounded. It should be clear that there are many functions which do not belong to $\mathfrak{B}(X, Y)$ and hence that there are many functions upon which \bar{d} cannot be applied. For one example, let $f : (0, 1) \rightarrow \mathbb{R}$ be given by $f(x) = \frac{1}{x}$ and $g(x) = 0$.

We saw in chapter 4 that a fundamental property of a metric space is its completeness. What can be said about the completeness of (\mathfrak{B}, \bar{d}) ?

Theorem 449 Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is complete, then the metric space (\mathfrak{B}, \bar{d}) of all bounded functions $f : X \rightarrow Y$ with sup norm \bar{d} is complete.

Exercise 6.0.2 Prove Theorem 449. Hint: To prove the theorem, the following method of constructing f is useful. Let $\langle f_n \rangle$ be a Cauchy sequence in (\mathfrak{B}, \bar{d}) . Then $\forall x \in X$, the sequence $\langle f_n(x) \rangle$ is Cauchy in Y and since (Y, d_Y) is complete we have $\langle f_n(x) \rangle$ converges to $f(x)$ in Y , $\forall x \in X$. Show that $f : X \rightarrow Y$ defined by $\lim_{n \rightarrow \infty} f_n(x), \forall x \in X$, is the function that $\langle f_n \rangle$ converges to with respect to the sup norm.

If Y is a vector space, then the metric space (Y, d_Y) is also a normed vector space by Theorem 207. Then $\mathcal{F}(X, Y)$ is a vector space where $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$. Its subspace of all bounded functions $\mathfrak{B}(X, Y) \subset \mathcal{F}(X, Y)$ is a normed vector space $(\mathfrak{B}, \|\cdot\|)$ with the norm $\|f\| = \bar{d}(f, 0) = \sup\{\|f(x)\|, x \in X\}$. This norm is called the sup norm. Theorem 449 states that if $(Y, \|\cdot\|_Y)$ is complete, then $(\mathfrak{B}, \|\cdot\|)$ is also complete.

A sequence $\langle f_n \rangle$ converges to $f : X \rightarrow Y$ in \mathfrak{B} with respect to the sup norm if

$$\|f_n - f\| = \sup \{\|f_n(x) - f(x)\|, x \in X\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In Chapter 4 we introduced two types of convergence of a sequence of functions: pointwise and uniform. These two types of convergence are defined in terms of the metric (norm) of the space Y (e.g. if $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, then in terms of the absolute value). Now we introduced (\mathfrak{B}, \bar{d}) where \bar{d} is the metric on $\mathfrak{B}(X, Y)$. As in any metric space we define convergence of elements (in this case functions) with respect to its metric (in this case \bar{d}). The question is, "Is convergence with respect to \bar{d} related to pointwise or uniform convergence respectively?" The following theorem addresses this question.

Theorem 450 *Let $\langle f_n \rangle$ be a sequence of functions in $\mathfrak{B}(X, Y)$. Then $\langle f_n \rangle \rightarrow f \in \mathfrak{B}(X, Y)$ with respect to the sup norm if and only if $\langle f_n \rangle \rightarrow f$ uniformly on X .*

Exercise 6.0.3 *Prove Theorem 450.*

In light of Theorem 450, one might wonder if there exists a metric d on $\mathcal{F}(X, Y)$ or on a subspace such that convergence of $\langle f_n \rangle$ with respect to this metric would be equivalent to pointwise convergence. Unfortunately, no such metric exists.

Before proceeding, we list the principal results of the chapter. Here we introduce two important function spaces: the space of bounded continuous functions (denoted $\mathcal{C}(X)$) and p -integrable functions (denoted $L_p(X)$). We give necessary and sufficient conditions for compactness in $\mathcal{C}(X)$ in Ascoli's Theorem 458. Then we deal with the problem of approximating continuous functions. The fundamental result is given in a very general set of Theorems by Stone and Weierstrass (the lattice version is 464 and the algebraic version is 468) which provide the conditions for a set to be dense in $\mathcal{C}(X)$. Next the Brouwer Fixed Point Theorem 302 of Chapter 4 on finite dimensional spaces is generalized to infinite dimensional Banach spaces in the Schauder Fixed Point Theorem 475. Next we introduce the $L_p(X)$ space and show that it is complete in the Riesz-Fischer Theorem 481. Among L_p spaces, we show that L_2 is a Hilbert space (i.e. that it is a complete normed vector space with the inner product) and consider the Fourier series of a function in L_2 . Then we introduce linear operators and functionals, as well as the notion of

a dual space of a normed vector space. We construct dual spaces for most common spaces: Euclidean, Hilbert, ℓ_p , and most importantly for L_p in the Riesz Representation Theorem 532. Next we show that one can construct bounded linear functionals on a given space X in the Hahn-Banach Theorem 539 which is used to prove certain separation results such as the fact that two disjoint convex sets can be separated by a linear functional in Theorem 549. Such results are used extensively in economics; for instance, it is employed to establish the Second Welfare Theorem. The chapter ends with nonlinear operators. First we introduce the weak topology on a normed vector space and develop a variational method of optimizing nonlinear functions in Theorem 572. Then we consider another method of finding the optimum of a nonlinear functional by dynamic programming.

6.1 The set of bounded continuous functions

Let $\mathcal{C}(X, Y)$ denote the set of all continuous functions $f : X \rightarrow Y$. In order to define a normed vector space, we need to equip this set with a norm. We first consider the sup norm. Since a continuous function can be unbounded (e.g. $f : (0, 1] \rightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x}$), the sup norm may not be well defined on the whole set $\mathcal{C}(X, Y)$. Hence we will restrict attention to a subset of $\mathcal{C}(X, Y)$ that contains only bounded continuous functions, which we denote $\mathcal{BC}(X, Y)$. Next we consider important properties of this space $(\mathcal{BC}(X, Y), \|\cdot\|_\infty)$ where $\|\cdot\|_\infty$ is the sup norm.

6.1.1 Completeness

Even if (Y, d_Y) is complete, we cannot directly use Theorem 449 to prove that $(\mathcal{BC}(X, Y), \|\cdot\|_\infty)$ is complete because $(\mathcal{BC}(X, Y), \|\cdot\|_\infty)$ is a subspace of the complete space $(\mathfrak{B}(X, Y), \|\cdot\|_\infty)$. But if we show that $\mathcal{BC}(X, Y)$ is closed in $\mathfrak{B}(X, Y)$, then the fact that a closed subspace of a complete space is complete by Theorem 177 in Chapter 4 would imply that $(\mathcal{BC}(X, Y), \|\cdot\|_\infty)$ is complete.

Lemma 451 *$\mathcal{BC}(X, Y)$ is closed in $\mathfrak{B}(X, Y)$; that is, if a sequence $\langle f_n \rangle$ of functions from $\mathcal{BC}(X, Y)$ converges to a function $f : X \rightarrow Y$, then f is continuous.*

Proof. (Sketch) We need to show that f is continuous at any $x_0 \in X$ (i.e. if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in X$ with $d_X(x, x_0) < \delta$ we have $d_Y(f(x), f(x_0)) < \varepsilon$). By the triangle inequality

$$d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(x_0)) + d_Y(f_n(x_0), f(x_0)).$$

The first and third terms on the right hand side are arbitrarily small (i.e. $\frac{\varepsilon}{4}$) since $f_n \rightarrow f$ with respect to the sup norm and the second term is arbitrarily small ($\frac{\varepsilon}{2}$) since f_n is continuous. ■

The next theorem establishes that $(\mathcal{BC}(X, Y), \|\cdot\|_\infty)$ is a Banach space.

Theorem 452 *The normed vector space $(\mathcal{BC}(X, Y), \|\cdot\|_\infty)$ is complete.*

Proof. Follows from Lemma 451 and Theorem 177. That $(\mathcal{BC}(X, Y), \|\cdot\|_\infty)$ is a Banach space follows from Definition 208. ■

In the remainder of this section, we will assume that X is a compact set and $(Y, d_Y) = (\mathbb{R}, |\cdot|)$. In this case $f \in \mathcal{C}(X, \mathbb{R})$ is bounded by Theorem 261. Hence, instead of $(\mathcal{BC}(X, \mathbb{R}), \|\cdot\|_\infty)$ we will simply use the notation $\mathcal{C}(X)$. Just remember, whenever you see $\mathcal{C}(X)$ we are assuming that X is compact, Y is \mathbb{R} , and we are considering the sup norm.

While uniform convergence implies pointwise convergence, we know the the converse does not hold (e.g. $f_n : [0, 1] \rightarrow \mathbb{R}$ given by $f_n(x) = x^n$). In $\mathcal{C}(X)$, however, there is a sufficient condition for uniform convergence (and hence for convergence with respect to the sup norm) in terms of pointwise convergence.

Lemma 453 (Dini's Theorem) *Let $\langle f_n \rangle$ be a monotone sequence in $\mathcal{C}(X)$ (e.g. $f_{n+1} \leq f_n, \forall n$). If the sequence $\langle f_n \rangle$ converges pointwise to a continuous function $f \in \mathcal{C}(X)$, it also converges uniformly to f .*

Proof. (Sketch) Let $\langle f_n \rangle$ be decreasing, $f_n \rightarrow f$ pointwise, and define $\bar{f}_n = f_n - f$. Then $\langle \bar{f}_n \rangle$ is a decreasing sequence of non-negative functions with $\bar{f}_n \rightarrow 0$ pointwise. Given $\varepsilon > 0$, for each $x \in X$, pointwise convergence guarantees we can find an index $N(\varepsilon, x)$ for which $0 \leq \bar{f}_{N(\varepsilon, x)}(x) < \varepsilon$. Due to continuity of $\bar{f}_{N(\varepsilon, x)}$ there is a $\delta(x)$ neighborhood around x such that $0 \leq \bar{f}_{N(\varepsilon, x)}(x') < \varepsilon$ for each x' of this neighborhood and due to monotonicity of $\langle \bar{f}_n \rangle$ we have $0 \leq \bar{f}_n(x') < \varepsilon$ for $n \geq N(\varepsilon, x)$. Since X is compact, there are finitely many points $x_i \in X$ whose neighborhoods $B_{\delta(x_i)}(x_i)$ cover X .

From finitely many corresponding indices $N(\varepsilon, x_i)$ we can find the minimum $N(\varepsilon)$ required for uniform convergence. ■

It is clear from the above example $f_n(x) = x^n$ that the requirement that f be continuous is essential. In the above case, f is clearly not continuous since

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}.$$

6.1.2 Compactness

While Theorem 193 established (necessary and) sufficient conditions (completeness and total boundedness) for compactness in a general metric space (and hence a general normed vector space), total boundedness was difficult to establish. As in the case of the Heine-Borel Theorem 194 (which provided simple sufficient conditions for a set in \mathbb{R}^n to be compact), here we develop the notion of equicontinuity which will be included as a sufficient condition for compactness.

Definition 454 *Let (X, d_X) and (Y, d_Y) be metric spaces. Let \mathcal{D} be a subset of the function space $\mathcal{BC}(X, Y)$. If $x_0 \in X$, the set \mathcal{D} of functions is **equicontinuous** at x_0 if $\forall \varepsilon > 0, \exists \delta(x_0, \varepsilon)$ such that $\forall x \in X, d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \varepsilon, \forall f \in \mathcal{D}$. If the set \mathcal{D} is equicontinuous at x_0 for each $x_0 \in X$, then it is equicontinuous on X .*

Notice that the primary difference between the definition of equicontinuity and that of continuity in 244 is that here $d_Y(f(x), f(x_0)) < \varepsilon$ must hold for all $f \in \mathcal{D}$, while in the former this condition must hold only for the given function f .

Example 455 *Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be given by $f_n(x) = x^n$ and $\mathcal{D} = \{f_n\}$. At what points is \mathcal{D} equicontinuous and at what points does it fail to be equicontinuous? It fails at $x = 1$. To see this, let $x_0 = 1$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $x \in [0, 1]$ with $d_X(x, 1) < \delta$ we have $|f_N(x) - f_N(1)| = 1 - x^N \geq \varepsilon$. Take the logs of both sides of $1 - \varepsilon \geq x^N$ and notice that $\log(x) < 0$ on $[0, 1]$ to yield $n \geq \frac{\ln(1-\varepsilon)}{\ln(x)}$ (by the Archimedean property such an n exists) so that we can take $N = w \left(\frac{\ln(1-\varepsilon)}{\ln(x)} \right) + 1$.*

In general δ in Definition 454 depends on both ε and x . If, however, the choice of δ is independent of x we say that the set of functions \mathcal{D} is *uniformly*

equicontinuous on X if $\forall \varepsilon > 0, \exists \delta(\varepsilon)$ such that $\forall x, x' \in X, d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon, \forall f \in \mathcal{D}$.

If X is compact, then these two notions are equivalent as the following lemma shows.¹

Lemma 456 *Let X be compact. A subset $\mathcal{D} \subset \mathcal{C}(X)$ is equicontinuous iff it is uniformly equicontinuous.*

Proof. (Sketch) (\Rightarrow) Since $\mathcal{D} \subset \mathcal{C}(X)$ is equicontinuous at $x \in X$, then given ε we can find $\delta(\varepsilon, x)$. Then the collection $\{B_{\delta(\varepsilon, x)}(x), x \in X\}$ covers X and because X is compact, there exists a finite subcollection covering X and a corresponding finite set of $\{\delta(\varepsilon, x_i), i = 1, \dots, k\}$. Then there exists a smallest $\delta(\varepsilon)$ that doesn't depend on x . ■

Equicontinuity is related to total boundedness when both X and Y are compact as the following lemma shows.

Lemma 457 *Let X be compact and $Y \subset \mathbb{R}$ be compact. Let \mathcal{D} be a subset of $\mathcal{C}(X, Y)$. Then \mathcal{D} is equicontinuous iff \mathcal{D} is totally bounded in the sup norm.*

Proof. (Sketch) (\Leftarrow) Let \mathcal{D} be totally bounded. Given $\varepsilon > 0$, choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $2\varepsilon_1 + \varepsilon_2 < \varepsilon$. Then for given ε_1 , there are finitely many functions $\{f_i, i = 1, \dots, k\}$ such that ε_1 balls around them cover \mathcal{D} . Since any finite collection of continuous functions is equicontinuous (see Exercise 6.1.1), given x_0 and ε_2 , there exists δ such that if $d_X(x, x_0) < \delta$, then $d_Y(f_i(x), f_i(x_0)) < \varepsilon_2$ for $i = 1, \dots, k$. We make a similar “estimate” for any $f \in \mathcal{D}$. But because there is an f_i which is “ ε_1 -close” to f , then we split $d_Y(f(x), f(x_0))$ into three parts (using the triangle inequality)

$$\begin{aligned} d_Y(f(x), f(x_0)) &\leq d_Y(f(x), f_i(x)) + d_Y(f_i(x), f_i(x_0)) + d_Y(f_i(x_0), f(x_0)) \\ &\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_1 < \varepsilon. \end{aligned}$$

The first and third terms are sufficiently small because f_i is “ ε_1 -close” to f . The second term is sufficiently small because $\{f_i, i = 1, \dots, k\}$ is equicontinuous.²

¹This lemma is similar to the result that a continuous function on a compact set is uniformly continuous.

²Notice that we haven't used compactness of X nor Y in this direction. Thus total boundedness always implies equicontinuity.

(\Rightarrow) Since X is compact and \mathcal{D} is equicontinuous, then by Lemma 456 \mathcal{D} is uniformly equicontinuous. Then given ε_1 there exists $\delta(\varepsilon_1)$ and finitely many points $\{x_i \in X, i = 1, \dots, k\}$ such that $\{B_{\delta(\varepsilon_1)}(x_i)\}$ covers X and $d_Y(f(x_i), f(x)) < \varepsilon_1$ for $x \in B_{\delta(\varepsilon_1)}(x_i)$ for all $f \in \mathcal{D}$. Since Y is compact, then Y is totally bounded by Theorem 198. Then given ε_2 there exist finitely many points $\{y_i \in Y, i = 1, \dots, m\}$ such that $\{B_{\varepsilon_2}(y_i), i = 1, \dots, m\}$ covers Y . Let J be the set of all functions α such that $\alpha : \{1, \dots, k\} \rightarrow \{1, \dots, m\}$. The set J is finite (it contains m^k elements). For $\alpha \in J$, choose $f \in \mathcal{D}$ such that $f(x_i) \in B_{\varepsilon_2}(y_{\alpha(i)})$ and label it f_α (for example, the index α of the function f_α in Figure 6.1.1 is $\alpha(1, 2, 3, 4) = (2, 3, 1, 1)$ because $f(x_1) \in B_{\varepsilon_2}(y_2)$, $f(x_2) \in B_{\varepsilon_2}(y_3)$, $f(x_3) \in B_{\varepsilon_2}(y_1)$, $f(x_4) \in B_{\varepsilon_2}(y_1)$). Then the collection of open balls $\{B_\varepsilon(f_\alpha), \alpha \in J\}$ with $\varepsilon \leq 2\varepsilon_1 + \varepsilon_2$ is a finite ε -covering of \mathcal{D} . Let $f \in \mathcal{D}$. Then $f(x_i) \in B_{\varepsilon_2}(y_{\alpha(i)})$ for $i = 1, \dots, k$. Choose this index α and corresponding f_α . Then one must show that $f \in B_\varepsilon(f_\alpha)$. Let $x \in X$. Then there exists i such that $x \in B_{\delta(\varepsilon_1)}(x_i)$ where

$$\begin{aligned} d_Y(f(x), f_\alpha(x)) &\leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f_\alpha(x)) + d_Y(f_\alpha(x_i), f_\alpha(x)) \\ &\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_1 \leq \varepsilon. \end{aligned}$$

The first and third terms are sufficiently small because \mathcal{D} is uniformly equicontinuous and the second term is sufficiently small because $f(x_i), f_\alpha(x_i) \in B_{\varepsilon_2}(y_{\alpha(i)})$. ■

Exercise 6.1.1 Show that a set which contains finitely many continuous functions is equicontinuous. *Hint: Since the collection of f_i is finite, there are finitely many δ_i associated with each one and hence the minimum of those δ_i is well defined.*

Thus $d_Y(f(x), f_{j(i)}(x)) < \varepsilon$ holds for any $x \in X$. Hence there are finitely many open balls $\{B_\varepsilon(f_{j(i)}), i = 1, \dots, k\}$ covering \mathcal{D} . Before stating the main theorem of this subsection, we point out something about boundedness in $\mathcal{C}(X)$. In a normed vector space $(X, \|\cdot\|)$ a subset \mathcal{A} is said to be bounded if it is contained in a ball (i.e. $\exists M$ such that $\|f\| \leq M, \forall f \in \mathcal{A}$). Since in $\mathcal{C}(X)$ we have $\|f\| = \sup_x |f(x)|$, this is equivalent to $\exists M$ such that $|f(x)| \leq M, \forall f \in \mathcal{D}, \forall x \in X$. This is sometimes called uniform boundedness of a set of functions \mathcal{D} . However, in terms of the normed vector space $\mathcal{C}(X)$ it is just the normal definition of boundedness.

Analogous to the Heine-Borel theorem in \mathbb{R} , we now state necessary and sufficient conditions for compactness in $\mathcal{C}(X)$.

Theorem 458 (Ascoli) *Let X be a compact space. A subset \mathcal{D} of $\mathcal{C}(X)$ is compact iff it is closed, bounded, and equicontinuous.*

Proof. Step 1: If \mathcal{D} is bounded, then $|f(x)| \leq M, \forall f \in \mathcal{A}, \forall x \in X$. Then \mathcal{D} is a subset of the ball $B_M(0)$. Let Y be the closure of $B_M(0)$. Then Y is a closed, bounded subset of \mathbb{R} and hence compact. Then $\mathcal{D} \subset \mathcal{C}(X, Y)$ with both X and Y compact.

Step 2: (\Rightarrow) Suppose $\mathcal{D} \subset \mathcal{C}(X, Y)$ is compact. Then by Lemma 189, \mathcal{D} is closed. By Lemma 187, \mathcal{D} is bounded. By Theorem 198, \mathcal{D} is totally bounded. But with X and Y compact (step 1) total boundedness is equivalent to equicontinuity.

Step 3: (\Leftarrow) Suppose \mathcal{D} is closed, bounded and equicontinuous (or totally bounded). By Theorem 177 a closed subset of a complete normed vector space $\mathcal{C}(X)$ is complete. By Theorem 198, completeness and total boundedness is equivalent to compactness. ■

Example 459 *Is the unit ball in $\mathcal{C}(X)$ a compact set? Without loss of generality we can take $X = [0, 1]$. The unit ball B_1 in $\mathcal{C}([0, 1])$ is $B_1(0) = \{f \in \mathcal{C}([0, 1]) : \|f\| \leq 1\}$. $B_1(0)$ is clearly bounded and closed. Is it equicontinuous? In Example 455 we showed that $\{x^n, n \in \mathbb{N}\}$ was not equicontinuous. But since $\|x^n\| = \sup_{x \in [0, 1]} |x^n| = 1$ for each $n \in \mathbb{N}$, then $\{x^n, n \in \mathbb{N}\} \subset B_1(0)$. Thus $B_1(0)$ contains a subset which is not equicontinuous so that $B_1(0)$ is not equicontinuous. Then by Ascoli's Theorem 458, $B_1(0)$ is not compact.*

In the previous example, how can the unit ball be closed if it contains a sequence $\langle x^n \rangle$ converging to a function that doesn't belong to $B_1(0)$? It is because $\langle x^n \rangle$ is not convergent in $\mathcal{C}([0, 1])$.

6.1.3 Approximation

For many applications it is convenient to approximate continuous functions by functions of an elementary nature (e.g. functions which are piecewise linear or polynomials).

Definition 460 *Let $f \in \mathcal{F}(X, Y)$ with norm $\|\cdot\|_{\mathcal{F}}$. Given $\varepsilon > 0$, we say g (ε -)approximates f on X with respect to $\|\cdot\|_{\mathcal{F}}$ if $\|f - g\|_{\mathcal{F}} < \varepsilon$. If $f \in \mathcal{C}(X)$, then since we are using the sup norm, this is equivalent to $\forall \varepsilon > 0, \sup_{x \in X} \{|f(x) - g(x)|\} < \varepsilon$ in which case it is clear that the approximation is uniform. See Figure 6.1.2.*

The concept of approximation can be stated in terms of dense sets. Let \mathcal{H} be a subset of $\mathcal{C}(X)$. Recall from Definition 153 that \mathcal{H} is dense in $\mathcal{C}(X)$ if the closure of \mathcal{H} , denoted $\overline{\mathcal{H}}$, satisfies $\overline{\mathcal{H}} = \mathcal{C}(X)$. But by Theorem 148, $f \in \overline{\mathcal{H}}$ iff for any $\varepsilon > 0$, there exists $g \in \mathcal{H}$ such that $\|f - g\| < \varepsilon$. In other words, a function $f \in \mathcal{C}$ can be approximated by a function $g \in \mathcal{H} \subset \mathcal{C}(X)$ if $\overline{\mathcal{H}} = \mathcal{C}(X)$.

An alternative way to see this, is suppose we are trying to approximate a continuous function $f : X \rightarrow \mathbb{R}$ with X compact and suppose we know that the set of all polynomials is dense (we will prove this later). Then we could think about starting with a large degree of approximation error (say $\varepsilon_1 = 1$) and ask what polynomial function (call it $P_{1,n}(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$) bounds the error within ε_1 (i.e. $\|P_{1,n}(x) - f(x)\| < \varepsilon_1$). If this error is too large, we could choose a smaller one, say $\varepsilon_2 (= \frac{1}{2}) < \varepsilon_1$ and look for another polynomial function $P_{2,n}(x)$ such that $\|P_{2,n}(x) - f(x)\| < \varepsilon_2$. We could let $\mathcal{H} = \{P_1, P_2, \dots\} \subsetneq \mathcal{C}(X)$. Approximation is essentially constructing a sequence of polynomials $\langle P_n \rangle$ that converges to f with respect to the sup norm (i.e. uniformly).

The most general approximation theorem is known as the Stone-Weierstrass Theorem which provides conditions under which a vector subspace of $\mathcal{C}(X)$ is dense in $\mathcal{C}(X)$. There are two versions of this result: one uses lattices and the other is algebraic.

We begin by noting that the space $\mathcal{C}(X)$ has a lattice structure. If $f, g \in \mathcal{C}(X)$, so are the “meet” and “join” functions $f \wedge g$ and $f \vee g$ defined as $(f \wedge g)(x) = \min[f(x), g(x)]$ and $(f \vee g)(x) = \max[f(x), g(x)]$. To see that $f \wedge g$ and $f \vee g$ are continuous, note that

$$(f \wedge g)(x) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g| = \begin{cases} \frac{1}{2}(f + g) - \frac{1}{2}(f - g) & \text{if } f > g \\ \frac{1}{2}(f + g) + \frac{1}{2}(f - g) & \text{if } f < g \end{cases}$$

and

$$(f \vee g)(x) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|.$$

But linear combinations of continuous functions are continuous by Theorem 251. Recall from Definition 42, a subset \mathcal{H} of $\mathcal{C}(X)$ is a lattice if for every pair of functions $f, g \in \mathcal{H}$, we also have $f \wedge g$ and $f \vee g$ in \mathcal{H} .

Definition 461 A subset \mathcal{H} of $\mathcal{C}(X)$ is called *separating* (or \mathcal{H} separates points) if for any two distinct points $x, y \in X$, $\exists h \in \mathcal{H}$ with $h(x) \neq h(y)$.

Example 462 $H_1 = \{\text{all constant functions } f : X \rightarrow \mathbb{R}\}$ is a lattice but is not separating. To see this, let $f(x) = \kappa, \kappa \in \mathbb{R}$ be a constant function. Then H_1 is a totally ordered set (since for any two distinct elements κ_1 and κ_2 in \mathbb{R} we have, say, $\kappa_1 < \kappa_2$). Furthermore, these two elements have a maximum and a minimum. However, this set is not separating since $f(x) = f(y) = \kappa$ for $x \neq y$.

The lattice version of the Stone-Weierstrass theorem is the consequence of the following lemma.

Lemma 463 Suppose X has at least two elements. Let \mathcal{H} be a subset of $\mathcal{C}(X)$ satisfying: (i) \mathcal{H} is a lattice; (ii) Given $x_1, x_2 \in X, x_1 \neq x_2, \alpha_1, \alpha_2 \in \mathbb{R}$, there exists $h \in \mathcal{H}$ such that $h(x_1) = \alpha_1$ and $h(x_2) = \alpha_2$. Then \mathcal{H} is dense in $\mathcal{C}(X)$.

Proof. (Sketch) Take $f \in \mathcal{C}(X)$ and $\varepsilon > 0$. We want to find an element of \mathcal{H} that is within ε of f .

First, fix $x \in X$. By assumption (ii), $\forall y \neq x, \exists \eta_y \in \mathcal{H}$ such that $\eta_y(x) = f(x)$ and $\eta_y(y) = f(y)$. For $y \neq x$, set $O_y = \{x' \in X : \eta_y(x') > f(x') - \varepsilon\}$. This set is open since η_y and f are continuous and as y varies, $\cup_{y \neq x} O_y$ is an open covering of X . Since X is compact, there exists finitely many sets O_y such that $X = \cup_{j=1}^N O_{y_j}$ with $y_j \neq x, \forall j$. Then let $v_x = \max\{\eta_{y_1}, \dots, \eta_{y_N}\}$. Since \mathcal{H} is a lattice, $v_x \in \mathcal{H}$ with the same properties as the η_y 's; namely, $v_x(x) = f(x)$ and $v_x(x') > f(x') - \varepsilon, \forall x' \in X$.

Second, let x vary. For each $x \in X$, let $\Omega_x = \{x' \in X : v_x(x') < f(x') + \varepsilon\}$. By exactly the same argument as the first step, there exists finitely many sets $\Omega_{x_1}, \dots, \Omega_{x_j}$ covering X . Set $v = \min\{v_{x_1}, \dots, v_{x_j}\}$. Then $v \in \mathcal{H}$ and $f(x') - \varepsilon < v < f(x') + \varepsilon, \forall x' \in X$. This means $\|f - v\| \leq \varepsilon$. ■

Assumption (ii) in Lemma 463 appears hard to verify. But as we will show, if \mathcal{H} separates points in X and if \mathcal{H} contains all constant functions, then \mathcal{H} satisfies assumption (ii) of Lemma 463.

Theorem 464 (Stone-Weierstrass L) If: (i) \mathcal{H} is a separating vector subspace of $\mathcal{C}(X)$; (ii) \mathcal{H} is a lattice; (iii) \mathcal{H} contains all constant functions. Then \mathcal{H} is dense in $\mathcal{C}(X)$.

Proof. To apply the previous lemma, we must show that assumptions (i) and (iii) of the Theorem imply assumption (ii) of Lemma 463.

Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since \mathcal{H} is separating, $\exists h \in \mathcal{H}$ such that $h(x_1) \neq h(x_2)$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$, then the system of linear equations

$$\begin{aligned}\alpha_1 &= \mu + \lambda h(x_1) \\ \alpha_2 &= \mu + \lambda h(x_2)\end{aligned}$$

has a unique solution $(\mu, \lambda) \in \mathbb{R}^2$ since

$$\text{rank} \begin{bmatrix} h(x_1) & 1 \\ h(x_2) & 1 \end{bmatrix} = 2$$

because $h(x_1) \neq h(x_2)$. Set $g(x) = \mu + \lambda h(x)$. Since \mathcal{H} is a vector subspace containing constant functions, $g \in \mathcal{H}$. Moreover, we see that $g(x_1) = \alpha_1$ and $g(x_2) = \alpha_2$ so that assumption (ii) of Lemma 463 is satisfied. \blacksquare^3

The Stone-Weierstrass theorem is very general and covers many classes of elementary functions to approximate continuous functions. We now state the algebraic version of the Stone-Weierstrass theorem. Following this, we will apply whichever version is more suitable to some concrete examples.

Definition 465 We call a vector subspace $\mathcal{H} \subset \mathcal{C}(X)$ an **algebra of functions** (not to be confused with an algebra of sets) if it is closed under multiplication.

Hence $\mathcal{H} \subset \mathcal{C}(X)$ is an algebra of functions if: (i) $\forall f, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$, we have $\alpha f + \beta g \in \mathcal{H}$; (ii) $\forall f, g \in \mathcal{H}$, we have $f \cdot g \in \mathcal{H}$ (where $f \cdot g$ is defined as $(f \cdot g)(x) = f(x) \cdot g(x), \forall x \in X$).

Before stating the algebraic version of the Stone-Weierstrass Theorem, we prove the following set of lemmas.

Lemma 466 A vector subspace $\mathcal{H} \subset \mathcal{C}(X)$ is a lattice iff for every element $h \in \mathcal{H}$, the function $|h| \in \mathcal{H}$ as well.

Proof. (\Rightarrow) Let $h \in \mathcal{H}$. Then $|h| = \max(h, 0) - \min(h, 0)$ and since \mathcal{H} is a lattice as well as a vector subspace, then the r.h.s. is from \mathcal{H} . Thus $|h| \in \mathcal{H}$.

(\Leftarrow) We can write $\max(f, g) = \frac{1}{2} [(f + g) + \frac{1}{2}|f - g|]$ and $\min(f, g) = \frac{1}{2} [(f + g) - \frac{1}{2}|f - g|]$. The right hand sides hold since $f, g \in \mathcal{H}$, the absolute value is from \mathcal{H} , and \mathcal{H} is a vector space. \blacksquare

³Instead of assuming that \mathcal{H} contains all constants it is sufficient to assume that \mathcal{H} contains just the constant $c = 1$. Since \mathcal{H} is a vector space, it contains all scalar multiples of 1.

Note that Lemma 466 provides a very convenient way of checking that a subset of functions is a lattice. In the next lemma we construct a sequence of polynomials $\langle P_n \rangle$ converging uniformly to $|x|$ on $[-1, 1]$.

Lemma 467 *There exists a sequence of polynomials $\langle P_n \rangle$ that converges uniformly to $f(x) = |x|$ on $[-1, 1]$.*

Proof. (Sketch) We construct the sequence $\langle P_n(x) \rangle$ on $[-1, 1]$ by induction: for $n = 1$, $P_1(x) = 0$; given $P_n(x)$, we define $P_{n+1}(x) = P_n(x) + \frac{1}{2}(x^2 - P_n^2(x))$, $\forall n \in \mathbb{N}$. Then show that (i) $P_n(x) \leq P_{n+1}(x)$, $\forall x \in [-1, 1]$ (i.e. P_n is nondecreasing) and (ii) $P_n(x) \rightarrow |x|$ pointwise on $[-1, 1]$. Since the limit function $|x|$ is continuous, by Dini's Theorem 453 $\langle P_n(x) \rangle$ converges to $|x|$ uniformly on $[-1, 1]$. ■

Theorem 468 (Stone-Weierstrass A) *Every separating algebra of functions $\mathcal{H} \subset \mathcal{C}(X)$ containing all the constant functions is dense in $\mathcal{C}(X)$.*

Proof. If \mathcal{H} is a separating subalgebra of $\mathcal{C}(X)$ containing constant functions, then so is its closure $\overline{\mathcal{H}}$. Therefore it suffices to show that $\overline{\mathcal{H}}$ is a lattice and apply Theorem 464.

Let $f \in \overline{\mathcal{H}}$ be nonzero. By Lemma 467, $\exists \langle P_n \rangle$ of polynomials that converges uniformly on $[-1, 1]$ to $f(x) = |x|$. Since $-1 \leq \frac{f}{\|f\|} \leq 1$, the sequence of functions $\left\langle P_n \left(\frac{f}{\|f\|} \right) \right\rangle$ converges uniformly to $\left| \frac{f}{\|f\|} \right| = \frac{|f|}{\|f\|}$. But

$$\left\langle P_n \left(\frac{f}{\|f\|} \right) \right\rangle \rightarrow \frac{|f|}{\|f\|} \Leftrightarrow \left\langle \|f\| P_n \left(\frac{f}{\|f\|} \right) \right\rangle \rightarrow |f|.$$

Since $\overline{\mathcal{H}}$ is an algebra, all terms in this sequence are in $\overline{\mathcal{H}}$ (because an algebra is closed under linear combination and multiplication). Since $\overline{\mathcal{H}}$ is closed, $|f| \in \overline{\mathcal{H}}$. By Lemma 466, $\overline{\mathcal{H}}$ is a lattice. ■

Exercise 6.1.2 *Prove that if \mathcal{H} is a separating subalgebra of $\mathcal{C}(X)$, then $\overline{\mathcal{H}}$ is as well.*

Both versions of the Stone-Weierstrass Theorem are a very general statement about the density of a subset \mathcal{H} in $\mathcal{C}(X)$ or equivalently about approximation in $\mathcal{C}(X)$. As some of the next examples show, it covers all known approximation theorems of continuous functions.

Example 469 Let H_2 be the set of Lipschitz functions $h : X \rightarrow \mathbb{R}$ given by $|f(x) - f(y)| \leq Cd_X(x, y)$, $\forall x, y \in X$ and $C \in \mathbb{R}$. First, we must establish H_2 is a vector subspace of $\mathcal{C}(X)$ containing constant functions. That is we must establish H_2 is closed under addition and scalar multiplication. To see this, suppose $f, g \in H_2$ so that $|f(x) - f(y)| \leq C_1d_X(x, y)$ and $|g(x) - g(y)| \leq C_2d_X(x, y)$. Then

$$\begin{aligned} |(f + g)(x) - (f + g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq (C_1 + C_2)d_X(x, y) \end{aligned}$$

so that H_2 is closed under addition. Similarly, for $\alpha \in \mathbb{R}$,

$$|\alpha f(x) - \alpha f(y)| = |\alpha| |f(x) - f(y)| \leq |\alpha| C_1 d_X(x, y)$$

so that H_2 is closed under scalar multiplication. Second, we must establish that H_2 is a lattice. To see this, notice

$$||h(x)| - |h(y)|| \leq |h(x) - h(y)|$$

by the triangle inequality. Finally, we must establish that H_2 is separating. To see this, for $x \neq y$, the function $h(z) = d_X(x, z)$ is Lipschitz with constant 1 satisfying $h(x) = d_X(x, x) = 0$ and $h(y) = d_X(x, y) > 0$. Thus H_2 is dense in $\mathcal{C}(X)$ by Theorem 464.

Example 470 Let H_3 be the set of continuous piecewise linear functions $h : [a, b] \rightarrow \mathbb{R}$ given by $h(x) = b_k + a_k x$ for $c_{k-1} \leq x < c_k$, $k = 0, 1, \dots, n$ with $a = c_0 < c_1 < \dots < c_n = b$ and where $a_{k-1}c_k + b_{k-1} = a_k c_k + b_k$, $\forall k$ keeps h continuous. It is easy to show that H_3 : is a vector subspace of $\mathcal{C}([a, b])$ containing constant functions; is a lattice because $|g(x)| \in H_3$ iff $g \in H_3$; and is separating since $g(x) = x \in H_3$. Thus H_3 is dense in $\mathcal{C}([a, b])$.

Example 471 Let H_4 be the set of all polynomials $h : X \rightarrow \mathbb{R}$ where X is a compact subset of \mathbb{R}^n . It is easy to show that H_4 is a subalgebra of $\mathcal{C}(X)$ containing the constants and is separating. Thus H_4 is dense in $\mathcal{C}(X)$. To see H_4 is a subalgebra, note that if we multiply two polynomials, the product is still a polynomial.

A special case of Example 471 is $X = [a, b]$ known as the Weierstrass Approximation Theorem. Notice that all the previous examples guarantee

the existence of a dense set in $\mathcal{C}(X)$ but don't present a constructive method of approximating a continuous function. The next example shows how to find a sequence of polynomials $h_n(x; f)$ converging uniformly to $f(x)$ on $[0, 1]$.⁴

Example 472 Let H_5 be the set of Bernstein polynomials $b_n(f) : [0, 1] \rightarrow \mathbb{R}$ for a function $f : [0, 1] \rightarrow \mathbb{R}$ where

$$b_n(x; f) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}$$

with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$.

Example 473 Let H_6 be the set of all continuous functions differentiable to order $p = \infty$ on $X \subset \mathbb{R}^n$ (denoted $\mathcal{C}^\infty(X)$). It is easy to show that $\mathcal{C}^\infty(X)$ is a separating algebra containing constant functions. Thus $\mathcal{C}^\infty(X)$ is dense in $\mathcal{C}(X)$.

6.1.4 Separability of $\mathcal{C}(X)$

To see that $\mathcal{C}(X)$ is separable we must show that there exists a countable subset $S \subset \mathcal{C}(X)$ that is dense in $\mathcal{C}(X)$ (i.e. $\overline{S} = \mathcal{C}(X)$). Consider the set S of all polynomials defined on X with rational coefficients. From Example 471, we know that the set of all polynomials is dense in $\mathcal{C}(X)$. But any polynomial can be uniformly approximated by polynomials with rational coefficients since \mathbb{Q} is dense in \mathbb{R} .

Corollary 474 if X is compact, the set S of all polynomials in X with rational coefficients (which is a countable set) is dense in $\mathcal{C}(X)$. Hence, $\mathcal{C}(X)$ is separable.

6.1.5 Fixed point theorems

In Chapter 4 we proved Brouwer's fixed point Theorem 302 for continuous functions defined on a compact subset of \mathbb{R}^n . But this theorem holds true in a more general setting. In particular, we don't need to restrict it to a finite dimensional vector space; it can be extended to infinite dimensional vector spaces (i.e. function spaces).

⁴See Carothers p. 164 for a proof of uniform convergence.

Theorem 475 (Schauder) *Let K be a non-empty, compact, convex subset of a normed vector space and let $f : K \rightarrow K$ be a continuous function. Then f has a fixed point.*

Proof. (Sketch) Since K is compact, K is totally bounded. Hence, given $\varepsilon > 0$, there exists a finite set $\{y_i, i = 1, \dots, n\}$ such that the collection $\{B_\varepsilon(y_i), i = 1, \dots, n\}$ covers K . Let $K_\varepsilon = \text{co}(\{y_1, \dots, y_n\})$. Since K is convex and $y_i \in K$ for all $i = 1, \dots, n$, then $K_\varepsilon \subset K$ (by Exercise 4.5.4). Note that K_ε is finite dimensional and since it is also closed and bounded, it is compact (by Heine-Borel).

Define the “projection” function $p_\varepsilon : K \rightarrow K_\varepsilon$ by $P_\varepsilon(y) = \sum_{i=1}^n \theta_i(y)y_i$ such that the functions $\theta_i : K \rightarrow \mathbb{R}$ are continuous for $i = 1, \dots, n$, $\theta_i(y) \geq 0$, and $\sum_{i=1}^n \theta_i = 1$. The construction of θ_i is given in the proof in the appendix to this chapter. By construction, $P_\varepsilon(y)$ is an ε -approximation of y (i.e. $\|P_\varepsilon(y) - y\| < \varepsilon, \forall y \in K$). Now for the function $f : K \rightarrow K$ define $f_\varepsilon : K_\varepsilon \rightarrow K_\varepsilon$ by $f_\varepsilon(x) = P_\varepsilon(f(x))$, for all $x \in K_\varepsilon$. The function f_ε satisfies all the assumptions of Brouwer’s Fixed Point Theorem 302. Hence there exists $x_\varepsilon \in K_\varepsilon$ such that $x_\varepsilon = f_\varepsilon(x_\varepsilon)$.

Set $f(x_\varepsilon) = y_\varepsilon$ and choose a sequence $\langle \varepsilon_i \rangle$ converging to zero. We must show that the approximating sequence $\langle x_\varepsilon \rangle$ and $\langle y_\varepsilon \rangle$ converge to the same point. By construction $\langle y_{\varepsilon_i} \rangle$ is a sequence in K and since K is compact, there exists a convergent subsequence

$$\langle y_{g(\varepsilon_i)} \rangle \rightarrow \bar{y} \in K. \quad (6.1)$$

All that’s left is to show that $\langle x_\varepsilon \rangle = \langle f_\varepsilon(x_\varepsilon) \rangle$ also converges to \bar{y} .

$$\begin{aligned} \|x_\varepsilon - \bar{y}\| &= \|y_\varepsilon + x_\varepsilon - y_\varepsilon - \bar{y}\| = \|y_\varepsilon + f_\varepsilon(x_\varepsilon) - y_\varepsilon - \bar{y}\| \\ &= \|y_\varepsilon + P_\varepsilon(y_\varepsilon) - y_\varepsilon - \bar{y}\| \leq \|P_\varepsilon(y_\varepsilon) - y_\varepsilon\| + \|y_\varepsilon - \bar{y}\|. \end{aligned}$$

The first term is sufficiently small because $P_\varepsilon(y)$ approximates y and the second term is sufficiently small since $\langle y_\varepsilon \rangle \rightarrow \bar{y}$. Hence $\langle x_\varepsilon \rangle \rightarrow \bar{y}$. Since f is continuous, then $f(x_\varepsilon) \rightarrow f(\bar{y})$. Combining this and (6.1) we have $f(\bar{y}) = \bar{y}$ or that \bar{y} is a fixed point of f . ■

Schauder’s Fixed Point Theorem requires compactness of a subset K of the function space $\mathcal{C}(X)$. We will now state it in a slightly different form that is more suitable for applications in function spaces (i.e. the assumptions of the following theorem are easier to verify).

Theorem 476 *Let $F \subset \mathcal{C}(X)$ be a nonempty, closed, bounded, and convex set with X compact. If the mapping $T : F \rightarrow F$ is continuous and if the family $T(F)$ is equicontinuous, then T has a fixed point in F .*

Proof. $T(F) \subset F$ by assumption. Set $H = \text{co}(\overline{T(F)})$ (i.e. H is the convex hull of the closure of $T(F)$). By definition H is closed and convex. If we show that H is equicontinuous we are done since then by Ascoli's Theorem 458 H is compact and T is continuous. By Schauder's Theorem 475 $T : H \rightarrow H$ has a fixed point.

We need to show that if $T(F)$ is equicontinuous, then $\text{co}(\overline{T(F)})$ is equicontinuous. Let $f \in \text{co}(\overline{T(F)})$. Then $f = \sum_{i=1}^k \lambda_i f^i$ such that $f^i \in \overline{T(F)}$, $\lambda_i > 0$, and $\sum_{i=1}^k \lambda_i = 1$ (which obviously implies $\lambda_i \leq 1$ for $i = 1, \dots, k$). $f^i \in \overline{T(F)}$ implies that $\exists \langle f_n^i \rangle_{n=1}^\infty \rightarrow f^i$ where $f_n^i \in T(F)$. Since

$$\begin{aligned} \|f(x) - f(y)\| &= \left\| \sum_{i=1}^k \lambda_i f^i(x) - \sum_{i=1}^k \lambda_i f^i(y) \right\| \\ &\leq \sum_{i=1}^k \|f^i(x) - f^i(y)\| \\ &\leq \sum_{i=1}^k [\|f^i(x) - f_n^i(x)\| + \|f_n^i(x) - f_n^i(y)\| + \|f_n^i(y) - f^i(y)\|]. \end{aligned}$$

This expression is arbitrarily small for x, y close because it is the sum of finitely (k) many expressions which are arbitrarily small. In particular, the first and third terms are arbitrarily small because $\langle f_n^i \rangle \rightarrow f^i$ and the second term is sufficiently small because $f_n^i \in T(F)$ and $T(F)$ is uniformly equicontinuous. ■

6.2 Classical Banach spaces: L_p

In the previous section we analysed the space of all bounded continuous functions $f : X \rightarrow \mathbb{R}$ equipped with the (sup) norm $\|f\|_\infty = \sup\{|f(x)|, x \in X\}$. There we showed that $(\mathcal{BC}(X, \mathbb{R}), \|\cdot\|_\infty)$ is complete.

There are some potential problems using this normed vector space. Convergence with respect to the sup norm in the set $\mathcal{BC}(X, \mathbb{R})$ is uniform convergence (by Theorem 450), which is quite restrictive. For example, the

sequence $\langle f_n(x) \rangle = \langle x^n \rangle$ on $X = [0, 1]$ is not convergent in the space $\mathcal{C}([0, 1])$. That is, in Example 455 we showed that $\langle x^n \rangle$ does not converge uniformly. We also mentioned that a metric (and hence a norm) that would induce pointwise convergence does not exist.

Does there exist a norm on the set $\mathcal{C}([0, 1])$ for which $\langle x^n \rangle$ would be convergent? Since $x \in [0, 1]$, $\langle x^n \rangle$ is bounded and $x^n \rightarrow 0$ pointwise a.e. (i.e. except at $x = 1$). The sequence $\langle \int_{[0,1]} x^n \rangle$ also converges (to 0) since

$$\lim_{n \rightarrow \infty} \|x^n - 0\|_1 = \lim_{n \rightarrow \infty} \int_{[0,1]} x^n = \int \lim_{n \rightarrow \infty} x^n = \int_{[0,1]} 0 = 0$$

where the second equality follows from the Bounded Convergence Theorem 386. Thus $\langle x^n \rangle$ on $[0, 1]$ converges with respect to the norm $\|\cdot\|_1$ to $f = 0$.

While we have defined a norm on $\mathcal{C}(X)$ that does not require strong convergence restrictions on a given sequence, we must establish whether $\mathcal{C}(X)$ equipped with $\|\cdot\|_1$ is complete. The next example shows this is not the case.

Example 477 Take $\mathcal{C}([-1, 1])$ with $f_n : [-1, 1] \rightarrow \mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [-1, 0] \\ 1 - nx & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{if } x \in [\frac{1}{n}, 1] \end{cases} .$$

See Figure 6.2.1. The sequence $\langle f_n(x) \rangle$ is Cauchy. To see this, we must show $\|f_n(x) - f_m(x)\|_1 \rightarrow 0$, with $n \geq m$ and m sufficiently large. Is it convergent in $\mathcal{C}([-1, 1])$ with respect to the norm $\|\cdot\|_1$? Let $f(x)$ be its limit. Then we must show

$$\begin{aligned} \|f_n - f\|_1 &= \int_{[-1,1]} |f_n(x) - f(x)| dx \\ &= \int_{[-1,0]} |1 - f(x)| dx + \int_{(0, \frac{1}{n})} |1 - nx - f(x)| dx + \int_{[\frac{1}{n}, 1]} |0 - f(x)| dx \end{aligned}$$

vanishes as $n \rightarrow \infty$. Since all the integrands on the right hand side are nonnegative, so is each integral. Hence $\|f_n - f\|_1 \rightarrow 0$ would imply each integral on the right hand side approaches zero as $n \rightarrow \infty$. Consequently

$$\lim_{n \rightarrow \infty} \int_{[-1,0]} |1 - f(x)| dx = 0 \text{ and } \lim_{n \rightarrow \infty} \int_{[\frac{1}{n}, 1]} |0 - f(x)| dx = 0$$

implies

$$f(x) = \begin{cases} 1 & \text{if } x \in [-1, 0] \\ 0 & \text{if } x \in (0, 1] \end{cases} .$$

But then $f(x)$ is not continuous on $[-1, 1]$ and hence $f(x) \notin \mathcal{C}([-1, 1])$ so that $\langle f_n(x) \rangle$ is not convergent. This proves that $(\mathcal{C}([-1, 1]), \|\cdot\|_1)$ is not a complete normed vector space.

To summarize, we have seen that the function space $\mathcal{C}(X)$ can be equipped with two norms: the sup norm and $\|\cdot\|_{p=1}$. In the former case, $(\mathcal{C}(X), \|\cdot\|)$ is complete but in the latter case, $(\mathcal{C}(X), \|\cdot\|_1)$ is not complete. Now we will introduce the space of all \mathcal{L} -measurable functions $f : X \rightarrow \mathbb{R}$ that are p -integrable. We will show that this space, known as the L_p space, is the completion of $\mathcal{C}(X)$ with respect to the $\|\cdot\|_p$ norm (just as, for instance, \mathbb{R} was the completion of \mathbb{Q}). Consider, then, the measure space $(\mathbb{R}, \mathcal{L}, m)$ where \mathcal{L} is a σ -algebra of all Lebesgue measurable sets and m is the Lebesgue measure. While we will work here with $(\mathbb{R}, \mathcal{L}, m)$, it can be extended to more general measure spaces (X, \mathcal{X}, μ) .

Definition 478 For any $p \in [1, \infty)$, we define $L_p(X)$ with $X \subset \mathbb{R}$ to be the space of all \mathcal{L} -measurable functions $f : X \rightarrow \mathbb{R}$ such that $\int_X |f(x)|^p dx < \infty$ and $L_\infty(X)$ to be the space of all essentially bounded \mathcal{L} -measurable functions (i.e. functions which are bounded almost everywhere - See Figure 6.2.2) Furthermore, define the function $\|\cdot\|_p : L_p(X) \rightarrow \mathbb{R}$ as

$$\|f\|_p = \begin{cases} (\int_X |f|^p)^{\frac{1}{p}} & p \in [1, \infty) \\ \text{ess sup } |f| & p = \infty \end{cases} .$$

We shall establish that $\|\cdot\|_p$ defines a norm on $L_p(X)$. For $p \in [1, \infty)$, this norm is called the L_p -norm or simply the p -norm or Lebesgue norm. To show that $\|\cdot\|_p$ satisfies the triangle inequality property required of a norm, we use the same procedures as we used in ℓ_p spaces in Chapter 4.

Theorem 479 (Riesz-Holder Inequality) Let p, q be nonnegative conjugate real numbers (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). If $f \in L_p(X)$ and $g \in L_q(X)$, then $fg \in L_1$ and $\int_X |fg| \leq \|f\|_p \|g\|_q$, with equality iff $\alpha|f|^p = \beta|g|^q$ a.e. where α, β are nonzero constants.

Proof. When $p = 1$, then $q = \infty$. Take $f \in L_1$ and $g \in L_\infty$. Then g is bounded a.e. so that $|g| \leq M$ a.e. Now $|fg| \leq M|f|$ a.e. so that $fg \in L_1$. Integrating we have

$$\int_X |fg| \leq M \int_X |f| = \|f\|_1 \|g\|_\infty.$$

Next assume $p, q \in (1, \infty)$. If either $f = 0$ a.e. or $g = 0$ a.e., we have equality. Let $f \neq 0$ a.e. and $g \neq 0$ a.e. Substituting for $a = \frac{|f(x)|}{\|f\|_p}$ and $b = \frac{|g(x)|}{\|g\|_q}$ in Lemma 235, we have

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \left(\frac{1}{p}\right) \frac{|f(x)|^p}{(\|f\|_p)^p} + \left(\frac{1}{q}\right) \frac{|g(x)|^q}{(\|g\|_q)^q}.$$

Then $fg \in L_1$ and by integrating we get

$$\frac{\int_X |f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \left(\frac{1}{p}\right) \frac{\int_X |f(x)|^p}{(\|f\|_p)^p} + \left(\frac{1}{q}\right) \frac{\int_X |g(x)|^q}{(\|g\|_q)^q} = \frac{1}{p} + \frac{1}{q} = 1$$

or $\int fg \leq \|f\|_p \|g\|_q$. By Lemma 235 equality holds when $a^p = b^q$, which means $(\|g\|_q)^q |f|^p = (\|f\|_p)^p |g|^q$. ■

Now we need to show that $\|\cdot\|_p$ satisfies the triangle inequality property of a norm.

Theorem 480 (Riesz-Minkowski) For $p \in [1, \infty]$ and $f, g \in L_p$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. For $p = 1$ and $p = \infty$, it follows trivially from $|f + g| \leq |f| + |g|$. Let $p \in (1, \infty)$ and let $h = |f + g|^{p-1}$. Since $p-1 = \frac{p}{q}$, it follows that $h \in L_q$ and $(\|h\|_q)^q = \int |f + g|^p = (\|f + g\|_p)^p$. Now

$$\begin{aligned} (\|f + g\|_p)^p &= \int |f + g|^{1+\frac{p}{q}} = \int |f + g| |f + g|^{\frac{p}{q}} \\ &\leq \int |f|h + \int |g|h \\ &\leq (\|f\|_p + \|g\|_p) \|h\|_q \\ &= (\|f\|_p + \|g\|_p) (\|f + g\|_p)^{\frac{p}{q}} \end{aligned}$$

where the second inequality follows from Theorem 479. Since $p - \frac{p}{q} = 1$, dividing both sides by $(\|f + g\|_p)^{\frac{p}{q}} \neq 0$, we have $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. Finally, if $\|f + g\| = 0$, then the inequality holds trivially. ■

As in L_1 considered in section 5.4, we stress that the function $\|\cdot\|_p$ satisfies all properties of a norm except the zero property (i.e. $\|f\|_p = 0$ does not imply $f = 0$ everywhere). Using equivalence classes of functions rather than functions themselves, it can be shown as in the previous section that $\|f\|_p$ is a norm on L_p .

Again, the most important question we must ask of our new normed vector space is “Is it complete?” The next theorem provides the answer.

Theorem 481 (Riesz-Fischer) *For $p \in [1, \infty]$, $(L^p, \|\cdot\|_p)$ is a complete normed vector space (i.e. a Banach space).*

Proof. (Sketch) The proof for $p = 1$ was already given in Theorem 443. The proof for $p \in (1, \infty)$ is virtually identical. Finally, let $p = \infty$ and let $\langle f_n \rangle$ be a Cauchy sequence in L^∞ . For $x \in X$,

$$|f_k(x) - f_n(x)| \leq \|f_k - f_n\|_\infty \quad (6.2)$$

except on a set $A_{k,n} \subset X$ with $mA_{k,n} = 0$ by Definition 368 of the essential supremum. If $A = \cup_{k,n} A_{k,n}$, then $mA = 0$ and $|f_k(x) - f_n(x)| \leq \|f_k - f_n\|_\infty, \forall k, n \in \mathbb{N}$ with $k > n$ and $\forall x \in X \setminus A$. Since $\langle f_n(x) \rangle$ is Cauchy in \mathbb{R} , there exists a bounded function $f(x)$ that $\langle f_n(x) \rangle$ converges to $\forall x \in X \setminus A$. Moreover this convergence is uniform outside A as (6.2) indicates. ■

Now we would like to establish how L_p spaces are related to one another and also how they are related to the set of continuous functions $\mathcal{C}(X, Y)$. Before doing that, however, we present an example which shows that continuity does not guarantee that a function is an element of L_p .

Example 482 *Let $f : (0, 1) \rightarrow \mathbb{R}$ be given by $f(x) = \frac{1}{x}$. The function f is continuous on $(0, 1)$ but is not p -integrable for any p . Hence $f \in \mathcal{C}((0, 1), \mathbb{R})$ but $f \notin L_p((0, 1))$.*

Lemma 483 $\mathcal{BC}(X, \mathbb{R}) \subset L_\infty(X)$.

Proof. If a function is bounded and continuous, it must be essentially bounded. ■

Note that from Definition 478, if the function is continuous, then the ess sup is just the sup. That is, if $f \in \mathcal{BC}(X)$, then $\|f\|_\infty$ is the extension (in the sense of Definition 56) from $\mathcal{BC}(X)$ to $L_\infty(X)$ of the sup norm. This also justifies why we used the prior notation $\|\cdot\|_\infty$ for the sup norm. Thus convergence in $L_\infty(X)$ is equivalent to uniform convergence outside a set of measure zero. Note that if X is compact, then $\mathcal{C}(X) \subset L_\infty(X)$.

If we make additional assumptions about the domain X , however, there are inclusion relations among the $L_p(X)$ spaces and their associated norms.

Theorem 484 *If $m(X) < \infty$, then for $1 < p < q < \infty$ we have $L_\infty(X) \subset L_q(X) \subset L_p(X) \subset L_1(X)$ and $\|f\|_1 \leq c_1 \|f\|_p \leq c_2 \|f\|_q \leq c_3 \|f\|_\infty$ where c_i are constants which are independent of f .*

Proof. $L_\infty(X) \subset L_q(X)$ for $1 < q < \infty$ and $m(X) < \infty$ since if f is bounded a.e. (i.e. $f \in L_\infty$) and measurable, then since mX is finite we have that f is integrable by Theorem 382.

Assume $1 < p < q < \infty$. Let $f \in L_q$. Then $f^p \in L_{\frac{q}{p}}$. Set $\lambda = \frac{q}{p}$. Since $q > p$, we have $\lambda > 1$. Choose μ such that $\frac{1}{\lambda} + \frac{1}{\mu} = 1$. Then

$$\int |f|^p = \int |f|^p \cdot 1 \leq \left(\int |f|^{p\lambda} \right)^{\frac{1}{\lambda}} \cdot \left(\int 1 \right)^{\frac{1}{\mu}} = \left(\int |f|^q \right)^{\frac{p}{q}} \cdot (mX)^{\frac{1}{\mu}} < \infty$$

where the first inequality follows from Holder's Inequality (Theorem 479) and taking the p -th root of both sides of the above inequality we obtain $\|f\|_p \leq [m(X)]^{\frac{1}{p\mu}} \|f\|_q$. Hence $f \in L_p(X)$. ■

Note, for instance, the proof gives a constructive way to obtain the constant $c_2 = [m(X)]^{\frac{1}{p\mu}}$. Thus stating that for $p < q$, $L_q(X) \subset L_p(X)$ means that if f is q -integrable, then f is also p -integrable and $\|f\|_p \leq c \|f\|_q$ where c is some constant. This inequality implies that if a sequence $\langle f_n \rangle$ converges in $L_q(X)$, then $\langle f_n \rangle \subset L_p(X)$ and converges also in $L_p(X)$. Note also that in this theorem we compare normed vector spaces with different norms.

Putting the two previous lemmas together we have the following result.

Corollary 485 *If $m(X) < \infty$, $\mathcal{BC}(X) \subset L_p(X)$.*

Example 486 The inclusions in Theorem 484 are strict. For instance, let $1 \leq q < \infty$ and take $f : (0, 1) \rightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x^q}$. Then $f \in L_p((0, 1))$ for $p < q$ but $f \notin L_q((0, 1))$. In particular, take $f(x) = \frac{1}{\sqrt{x}}$. Then $\int_{(0,1)} \frac{1}{\sqrt{x}} dx = 2\sqrt{x}|_0^1 = 2$ so $f \in L_1((0, 1))$ but $\int_{(0,1)} \left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_{(0,1)} \frac{1}{x} dx = \ln(x)|_0^1 = \infty$ so $f \notin L_2((0, 1))$.

Example 487 The assumption that $m(X) < \infty$ is important. Take $f : [1, \infty) \rightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x^p}$ with $1 \leq p < \infty$. Then $f \in L_q([1, \infty))$ if $q > p$ but $f \notin L_p([1, \infty))$. In particular, $f(x) = \frac{1}{x} \in L_2([1, \infty))$ but $f \notin L_1([1, \infty))$.

Comparing Theorem 239 (in ℓ_p) and Theorem 484 (in L_p), one may wonder why the order of ℓ_p spaces is exactly opposite that of $L_p(X)$ spaces with $m(X) < \infty$. That is, for $1 < p < q < \infty$

$$\begin{aligned} \ell_1 &\subset \ell_p \subset \ell_q \subset \ell_\infty \\ L_1 &\supset L_p \supset L_q \supset L_\infty. \end{aligned}$$

ℓ_p spaces are spaces of sequences and we know that a sequence is just a function $f : \mathbb{N} \rightarrow \mathbb{R}$; that is, a function defined on an unbounded set. If $\langle x_i \rangle \in \ell_p$, then $\sum_{i=1}^{\infty} |x_i|^p < \infty$. This infinite sum can only be finite if $|x_i|^p$ decreases “rapidly enough” to zero. Now if $p < q$, then $|x_i|^q$ decreases “more rapidly” than $|x_i|^p$ (i.e. $|x_i|^q < |x_i|^p$). Hence if $\langle x_i \rangle \in \ell_p$, then $\langle x_i \rangle \in \ell_q$. In the case of $L_p(X)$ with $mX < \infty$, while X is bounded, $f : X \rightarrow \mathbb{R}$ may not be bounded. If $f \in L_q(X)$, then $\int_X |f|^q < \infty$. For $p < q$, $|f|^p < |f|^q \Rightarrow \int_X |f|^p < \int_X |f|^q < \infty$ and hence $f \in L_p(X)$.

6.2.1 Additional Topics in $L_p(X)$

Approximation in $L_p(X)$

For $L_p(X)$, $p \in (1, \infty)$ we have the following result which is similar to Theorem 445 in $L_1(X)$.

Theorem 488 Let $1 < p < \infty$, $X \subset \mathbb{R}$, $f \in L_p(X)$ and $\varepsilon > 0$. Then (i) there is an integrable simple function φ such that $\|f - \varphi\|_p < \varepsilon$; and (ii) there is a continuous function g such that g vanishes ($g = 0$) outside some bounded interval and such that $\|f - g\|_p < \varepsilon$.

Exercise 6.2.1 *Prove Theorem 488. Hint: See Carothers p.350.*

Note that here X can be equal to \mathbb{R} so that the theorem also covers sets of infinite measure.

Corollary 489 *The set of all integrable simple functions is dense in $L_p(X)$. The set of all continuous functions vanishing outside a bounded interval is dense in $L_p(X)$.*

Now consider the case where $p = \infty$. Let $f \in L_\infty(X)$ in which case f is bounded a.e. on X (i.e. there is a set E such that $m(E) = 0$ and f is bounded on $X \setminus E$). Then by Theorem 367, there exists a sequence of simple functions $\langle \varphi_n \rangle$ converging uniformly to f on $X \setminus E$. In other words, $\varphi_n \rightarrow f$ uniformly a.e. on X . Thus, $\varphi_n \rightarrow f$ in $L_\infty(X)$.

Corollary 490 *The simple functions are dense in $L_\infty(X)$.*

If $m(X) < \infty$, then any simple function is integrable. Thus we have:

Corollary 491 *If $m(X) < \infty$, then the integrable simple functions are dense in $L_\infty(X)$.*

Notice that the condition $m(X) < \infty$ is critical here. For example, $f = 1 \in L_\infty(\mathbb{R})$ cannot be approximated by an integrable simple function.

Separability of $L_p(X)$

If X is compact, then the set of all polynomials with rational coefficients $P_Q(X)$ is dense in $\mathcal{C}(X)$ and because $\mathcal{C}(X)$ is dense in $L_p(X)$ (by Corollary 474), then $P_Q(X)$ is also dense in $L_p(X)$. Thus $L_p(X)$ with X compact is separable.

If X is not compact, then as we showed in $L_1(X)$, the set M of all finite linear combinations of the form $\sum_{i=1}^n c_i \chi_{I_i}$ where c_i are rational numbers and I_i are intervals with rational endpoints is a countably dense set in $L_p(X)$.

Theorem 492 Corollary 493 *$L_p(X)$ is separable for $1 < p < \infty$.*

Corollary 494 *$L_\infty(X)$ is not separable for any X (either compact or not).*

Proof. Take two bounded functions $\chi_{[a,c]}$ and $\chi_{[a,d]}$. Since $\|\chi_{[a,c]} - \chi_{[a,d]}\|_\infty = 1$ for $c \neq d$, then

$$B_{\frac{1}{2}}(\chi_{[a,c]}) \cap B_{\frac{1}{2}}(\chi_{[a,d]}) = \emptyset \text{ for } c \neq d$$

where

$$B_{\frac{1}{2}}(\chi_{[a,c]}) = \left\{ f \in L_\infty : \|f - \chi_{[a,c]}\|_\infty < \frac{1}{2} \right\}.$$

Let F be an arbitrary set which is dense in $L_\infty([a, c])$. Then for each c with $a < c < b$ there is a function $f_c \in F$ such that $\|\chi_{[a,c]} - f_c\|_\infty < \frac{1}{2}$ since $\chi_{[a,c]} \in L_\infty(X)$ and F is dense in $L_\infty(X)$. Because $f_c \neq f_d$ for $c \neq d$ and there are uncountably many real numbers between $[a, b]$, F must be uncountable.

■

6.2.2 Hilbert Spaces ($L_2(X)$)

As we mentioned in Chapter 4, a Hilbert space is a Banach space equipped with an inner product. Hence, a Hilbert space is a special type of Banach space which possesses an additional structure: an inner product. This additional structure allows us, apart from measuring length of vectors (norms), to measure angles between vectors. In particular it enables us to introduce the notion of orthogonality for two vectors.

Definition 495 We say that two vectors x and y of M are **orthogonal** (perpendicular) if their inner product $\langle x, y \rangle = 0$ and we denote it $x \perp y$. The set $N \subset \mathcal{H}$ is called an **orthogonal set (or orthogonal system)** if any two different elements φ and ψ of N are orthogonal, that is $\langle \varphi, \psi \rangle = 0$. An orthogonal set N is called **orthonormal** if it is orthogonal and $\|\varphi\| = 1$ for each φ in N .

Example 496 \mathbb{R}^n with the inner product defined by $\langle x, y \rangle = x_1y_1 + \dots + x_ny_n = \sum_{i=1}^n x_iy_i$ is a Hilbert space. The set $N = \{e^i, i = 1, \dots, n\}$ where $e^i = (0, \dots, 0, 1, 0, \dots, 0)$ is orthonormal.

Example 497 ℓ_2 with the inner product defined by $\langle x, y \rangle = \sum_{i=1}^{\infty} x_iy_i$ where $x = \langle x_i \rangle$, $y = \langle y_i \rangle$ is a Hilbert space. The set $N = \{e^i, i \in \mathbb{N}\}$ is orthonormal.

Example 498 $L_2([0, 2\pi])$ with inner product defined by $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x) dx$ where $f, g \in L_2([0, 2\pi])$ is a Hilbert space. The set $N = \left\{ \frac{1}{\sqrt{2\pi}}, \cos \frac{nx}{\sqrt{\pi}}, \sin \frac{nx}{\sqrt{\pi}}, n \in \mathbb{N} \right\}$ is an orthonormal.

Exercise 6.2.2 Show that N in Example 498 is an orthonormal system.

Notice that the distance between any two distinct elements of an orthonormal system is $\sqrt{2}$. That is, $\|\varphi - \psi\|^2 = \langle (\varphi - \psi), (\varphi - \psi) \rangle = \langle \varphi, \varphi \rangle - \langle \varphi, \psi \rangle - \langle \psi, \varphi \rangle + \langle \psi, \psi \rangle = \|\varphi\|^2 + \|\psi\|^2 = 1 + 1 = 2$.

Lemma 499 If \mathcal{H} is a separable Hilbert space, then each orthonormal set is countable.

Proof. Let $U = \{e^\alpha, \alpha \in A\}$, where A is an index set, be an uncountable orthonormal set in \mathcal{H} . Then the collection of balls around each element e^α with radius $\frac{1}{2}$ (i.e. $\{B_{\frac{1}{2}}(e^\alpha), \alpha \in A\}$) would be an uncountable collection of disjoint balls and hence \mathcal{H} could not be separable. ■

Definition 500 An orthonormal set $\{e^\alpha, \alpha \in A\}$ is said to be **complete** if it is maximal.

In other words it is not possible to adjoin an additional element $e \in \mathcal{H}$ with $e \neq 0$ to $\{e^\alpha, \alpha \in A\}$ such that $\{e, e^\alpha, \alpha \in A\}$ is an orthonormal set in \mathcal{H} . The existence of a complete orthonormal set in any Hilbert space \mathcal{H} is guaranteed by Zorn's lemma because the collection $\{N\}$ of all orthonormal sets in \mathcal{H} is partially ordered by set inclusion. Thus we have the following.

Theorem 501 Every separable Hilbert space contains a countable complete orthonormal system.

The following theorem can be used to check if the orthonormal set is complete.

Theorem 502 $\{e^\alpha, \alpha \in A\}$ is a complete orthonormal set in \mathcal{H} iff $x \perp e^\alpha, \forall \alpha \in A$ implies $x = 0$.

Proof. By contradiction. (\Rightarrow) Let $\{e^\alpha, \forall \alpha \in A\}$ be complete and $\exists x \neq 0$ in \mathcal{H} such that $x \perp e^\alpha, \forall \alpha \in A$. Define $e = \frac{x}{\|x\|}$ so that $\|e\| = 1$. Hence $\{e, e^\alpha, \alpha \in A\}$ is orthonormal which contradicts the assumption that $\{e^\alpha, \alpha \in A\}$ is complete (maximal).

(\Leftarrow) Assume that $x \perp e^\alpha, \forall \alpha \in A \Rightarrow x = 0$ and $\{e^\alpha, \forall \alpha \in A\}$ is not complete. Then $\exists e \in \mathcal{H}$ such that $\{e, e^\alpha, \forall \alpha \in A\}$ is an orthonormal system and $e \notin \{e^\alpha, \forall \alpha \in A\}$. Since $e \perp e^\alpha, \forall \alpha \in A$ and $e \neq 0$ (because $\|e\| = 1$), the assumption is contradicted. ■

Exercise 6.2.3 Show that the orthonormal systems in \mathbb{R}^n, ℓ_2 , and $L_2([0, 2\pi])$ defined in examples 496 to 498 are complete.

Consider now a separable Hilbert space and let $\{e^i\}$ be an orthonormal system in \mathcal{H} . We know that $\{e^i\}$ is either a finite or countably infinite set. We define the Fourier coefficients with respect to $\{e^i\}$ of an element $x \in \mathcal{H}$ to be $a_i = \langle x, e^i \rangle$.

Theorem 503 (Bessel's Inequality) Let $\{e^i\}$ be an orthonormal system in \mathcal{H} and let $x \in \mathcal{H}$. Then

$$\sum_{i=1}^I a_i^2 \leq \|x\|^2$$

where $a_i = \langle x, e^i \rangle$ are the Fourier coefficients of x and $I = N$ if $\{e^i\}$ is finite or $I = \infty$ otherwise.

Proof. $0 \leq \|x - \sum_{i=1}^n a_i e^i\|^2 = \|x\|^2 - 2 \sum_{i=1}^n a_i \langle x, e^i \rangle + \sum_{i=1}^n \sum_{j=1}^n a_i a_j \langle e^i, e^j \rangle = \|x\|^2 - \sum_{i=1}^n a_i^2$. Thus $\sum_{i=1}^n a_i^2 \leq \|x\|^2$ and since n was arbitrary, we have $\sum_{i=1}^{\infty} a_i^2 \leq \|x\|^2$. ■

Now let a_i be Fourier coefficients of x with respect to $\{e^i\}$ and let $\sum_{i=1}^{\infty} a_i^2 < \infty$ (i.e. $\sum_{i=1}^{\infty} a_i^2$ converges). Then consider a sequence $\langle z \rangle$ defined by

$$z_n = \sum_{i=1}^n a_i e^i,$$

for $m \geq n$

$$z_m - z_n = \sum_{i=n+1}^m a_i e^i,$$

and we have $\|z_m - z_n\|^2 = \sum_{i=n}^m \sum_{j=n}^m a_i a_j \langle e^i, e^j \rangle = \sum_{i=n}^m a_i^2$. This term can be made sufficiently small for m, n large enough because $\sum_{i=1}^{\infty} a_i^2$ is convergent. Hence $\langle z_n \rangle$ is a Cauchy sequence. Because \mathcal{H} is a Hilbert space (thus complete), there exists $y \in \mathcal{H}$ such that $y = \sum_{i=1}^{\infty} a_i e^i$. Since the inner product is continuous, by the Cauchy-Schwartz inequality we have

$$\langle y, e^i \rangle = \lim_{n \rightarrow \infty} \langle z_n, e^i \rangle = \langle \sum_{j=1}^{\infty} a_j e^j, e^i \rangle = a_i.$$

Thus a_i are Fourier coefficients of y , as well as of x (which we started with). When does x equal y ? In other words, when are the elements with the same Fourier coefficients equal? Let a_i be the Fourier coefficients of two elements x and y (i.e. $a_i = \langle y, e^i \rangle = \langle x, e^i \rangle$). But this is equivalent to $0 = \langle y - x, e^i \rangle, \forall i = 1, 2, \dots$. This implies $x = y$ iff the orthonormal system $\{e^i\}$ is complete by Theorem 502. Hence we proved the following:

Theorem 504 (Parseval Equality) *If $\{e^i\}$ is a complete orthonormal system in a Hilbert space \mathcal{H} then for each $x \in \mathcal{H}$, $x = \sum_{i=1}^{\infty} a_i e^i$ (or $x = \sum_{i=1}^N a_i e^i$) where $a_i = \langle x, e^i \rangle$. Moreover $\|x\|^2 = \sum_{i=1}^{\infty} a_i^2$.*

To summarize the previous findings, let $\{e^i\}$ be a complete orthonormal system of a Hilbert space \mathcal{H} and let $a_i = \langle x, e^i \rangle, i = 1, 2, \dots$ be Fourier coefficients of x with respect to $\{e^i\}$. Then the Fourier series $\sum_{i=1}^{\infty} a_i e^i$ converges to x (with respect to the norm of \mathcal{H}). That is,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i e^i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \langle x, e^i \rangle e^i = x$$

or equivalently

$$\left\| \sum_{i=1}^n \langle x, e^i \rangle e^i - x \right\| \rightarrow 0.$$

This implies that if $x, y \in \mathcal{H}$ have the same Fourier coefficients, then $\|x - y\|_{\mathcal{H}} = 0$ which means $x = y$. Depending on the space we deal with this may mean that $x = y$ a.e.

Example 505 Add $L_2([0, 2\pi])$

6.3 Linear operators

In the previous two sections on $\mathcal{C}(X)$ and $L_p(X)$ we studied normed vector (linear) spaces whose elements were functions. In this section, we study functions that operate between two normed vector spaces. We call these functions operators (to distinguish them from the functions that are elements of the normed vector spaces).

We will focus primarily on operators that preserve the algebraic structure of vector (linear) spaces. These functions are called linear operators. Because normed vector spaces are also metric spaces, we will also address the issue of how linearity relates to continuity.

Definition 506 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. A function $T : X \rightarrow Y$ is called a **linear operator** if $T(\alpha x + \beta x') = \alpha T x + \beta T x'$, $\forall x, x' \in X$ and $\alpha, \beta \in \mathbb{R}$.

Example 507 Consider the Banach space $(\mathcal{C}([0, 1]), \|\cdot\|_\infty)$. Assume that a function $g : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is continuous. Define $T : \mathcal{C}([0, 1]) \rightarrow \mathcal{C}([0, 1])$ by $(Tx)(t) = \int_{[0,1]} g(t, s)x(s)ds$. For instance, $g(t, s)$ could be a joint density function and $x(s) = s$. Then $Tx(t)$ is the mean of s conditional on t . It is easy to show that T is linear (due to the linearity of the integral).

We would like to characterize continuous linear operators. First we prove an important fact about continuity of linear operators.

Theorem 508 Let X, Y be normed vector spaces and $T : X \rightarrow Y$ be a linear operator. Then T is continuous on X iff T is continuous at any one element in X .

Proof. (\Rightarrow) By definition.

(\Leftarrow) Let T be continuous at $x_0 \in X$ and $x \in X$ be arbitrary. Let $\langle x_n \rangle \subset X$ and $x_n \rightarrow x$. Then $\langle x_n - x + x_0 \rangle \rightarrow x_0$. Therefore, by Theorem 248, $T(x_n - x + x_0) \rightarrow T x_0$ (because T is continuous at x_0). But if $T(x_n - x + x_0) = T x_n - T x + T x_0 \rightarrow T x_0$ (where the equality follows from linearity of T), then $T x_n - T x \rightarrow 0 \Leftrightarrow T x_n \rightarrow T x$. Hence T is continuous at x . ■

Here we stress that all one needs to establish continuity is that T is continuity at one point. The result is a simple consequence of the linearity of the operator T (just as we proved in earlier chapters that a linear function is continuous). But one should not be confused; it is not the case that all

linear operators are continuous since it may not be continuous at any points in X (See Example 514).

Just as we considered restricting the space of all functions $\mathcal{F}(X, Y)$ from a metric space X to a metric space Y to the subset $\mathfrak{B}(X, Y)$ of all bounded functions in the introduction to this chapter, now we introduce a bounded linear operator and define a new norm.

Definition 509 *Let X, Y be normed vector spaces and $T : X \rightarrow Y$ be a linear operator. T is said to be **bounded** on X if $\exists K \in \mathbb{R}_{++}$ such that $\|Tx\|_Y \leq K \cdot \|x\|_X, \forall x \in X$.*

We note that this type of boundedness is different from that in Definition 163. In that case, we would say $\exists M$ such that $\|f(x)\|_Y \leq M, \forall x \in X$. The next example shows how different they are.

Example 510 *Let $(X, \|\cdot\|_X) = (\mathbb{R}, |\cdot|) = (Y, \|\cdot\|_Y)$ and $T : X \rightarrow Y$ be given by $Tx = 2x$. T as a linear function is not bounded on \mathbb{R} with respect to $\|\cdot\|_Y$ since $2x$ can be arbitrarily large. But T as a linear operator is bounded in the sense of Definition 509 since $\|Tx\|_Y \leq K \cdot \|x\|_X \Leftrightarrow |2x| \leq 2|x|, \forall x \in X$.*

In the remainder of the book, when we say that a linear operator is bounded, we mean it in the sense of Definition 509.

The following result shows that a bounded linear operator is equivalent to a continuous operator.

Theorem 511 *Let $T : X \rightarrow Y$ be a linear operator. Then T is continuous iff T is bounded.*

Proof. (\Leftarrow) Assume that T is bounded and let $\|x_n\|_X \rightarrow 0$. Then $\exists K$ such that $\|Tx_n\|_Y \leq K \|x_n\|_X \rightarrow 0$ as $n \rightarrow \infty$. But this implies $\|Tx_n\|_Y \rightarrow 0$ so that T is continuous at zero and hence continuous on X by Theorem 508.

(\Rightarrow) By contraposition. In particular, we will prove that if T is not bounded, then T is not continuous. If T is not bounded, then $\forall n \in \mathbb{N}$, $\exists x_n \in X$ with $x_n \neq 0$ such that $\|Tx_n\|_Y > n \|x_n\|_X$. But this implies

$$\left\| \frac{Tx_n}{n \|x_n\|_X} \right\|_Y > 1.$$

Setting $y_n = \frac{x_n}{n \|x_n\|_X}$, we know $\|y_n\|_X \rightarrow 0$ as $n \rightarrow \infty$. But $\|Ty_n\|_Y > 1, \forall n \in \mathbb{N}$. Thus Ty_n cannot converge to 0 and T is not continuous at 0 (and hence not continuous). ■

Example 512 Consider the linear operator $T : \mathcal{C}([0, 1]) \rightarrow \mathcal{C}([0, 1])$ defined in Example 507 by $(Tx)(t) = \int_{[0,1]} g(t, s)x(s)ds$. Since $g : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is a continuous function on a compact domain, it is bounded or $|g(x_1, x_2)| \leq M_1, \forall (x_1, x_2) \in [0, 1] \times [0, 1]$. Also, $x(t) : \mathcal{C}([0, 1]) \rightarrow \mathcal{C}([0, 1])$ is bounded by virtue of being in $\mathcal{C}([0, 1])$ or $|x(t)| \leq M_2, \forall t \in [0, 1]$. Thus,

$$(Tx)(t) = \int_{[0,1]} g(t, s)x(s)ds \leq M_1 \int_{[0,1]} |x(s)|ds \leq M_1 M_2.$$

Definition 513 Let $\mathfrak{L}(X, Y)$ be the set of all linear operators $T : X \rightarrow Y$ where X, Y are normed vector spaces. Let $\mathfrak{BL}(X, Y)$ be the set of all bounded linear operators in $\mathfrak{L}(X, Y)$.

The next example shows that $\mathfrak{BL}(X, Y)$ is a proper subset of $\mathfrak{L}(X, Y)$. Coupled with Theorem 511 it also shows that not all linear operators are continuous.

Example 514 Consider the normed vector space of all polynomials $P : [0, 1] \rightarrow \mathbb{R}$ with the sup norm $\|\cdot\|_\infty$. Define $T : P([0, 1]) \rightarrow P([0, 1])$ by $(Tx)(t) = \frac{dx(t)}{dt}, t \in [0, 1]$. T is called the differentiation operator. It is easy to check that T is linear (since the derivative of a sum is equal to the sum of the derivatives). But T is not bounded. To see why, let $\langle x_n(t) \rangle = t^n, \forall n \in \mathbb{N}$. Then $\|x_n\|_\infty = \sup\{|t^n|, t \in [0, 1]\} = 1$ and $(Tx_n)(t) = \frac{dx_n(t)}{dt} = n \cdot t^{n-1}$. Therefore, $\|Tx_n\|_\infty = \sup\{n|t|^{n-1}, t \in [0, 1]\} = n, \forall n \in \mathbb{N}$. Then T is not bounded since there is not a fixed number K such that $\frac{\|Tx_n\|_\infty}{\|x_n\|_\infty} = n \leq K$. The sequence of functions $x_n(t) = t^n$ converges to

$$x_0(t) = \begin{cases} 0 & t \in [0, 1) \\ 1 & t = 1 \end{cases}$$

but the sequence of their derivatives $x'_n(t) = nt^{n-1}$ doesn't converge to the derivative of $x_0(t)$ (which actually doesn't exist).

In the introduction to this Chapter we defined the sup norm on $\mathfrak{B}(X, Y) \subset \mathfrak{F}(X, Y)$. What would be the consequences of equipping $\mathfrak{BL}(X, Y)$ with the sup norm? More specifically, how large would $(\mathfrak{BL}(X, Y), \|\cdot\|_\infty)$ be? The next example shows it would be very, very small.

Example 515 Take $X = Y = \mathbb{R}$.⁵ All linear functions $f : \mathbb{R} \rightarrow \mathbb{R}$ are of the form $Tx = ax$ but these are not bounded with respect to the sup norm. Hence the only element that would belong to $\mathfrak{BL}(X, Y)$ of all bounded linear operators equipped with the sup norm would be $Tx = 0, \forall x \in \mathbb{R}$.

Definition 516 Let $T \in \mathfrak{BL}(X, Y)$. Then T is bounded by assumption so $\exists K$ such that $\|Tx\|_Y \leq K \cdot \|x\|_X, \forall x \in X$. We call the least such K the **(operator) norm** of T and denote it $\|T\|$ where

$$\|T\| = \inf \{K : K > 0 \text{ and } \|Tx\|_Y \leq K \|x\|_X, x \in X\}. \quad (6.3)$$

Exercise 6.3.1 Prove that the function $\|T\|$ in (6.3) is a norm on $\mathfrak{BL}(X, Y)$.

What is the relation between the sup norm and this new operator norm? In the introduction to this Chapter we defined the sup norm on $\mathfrak{B}(X, Y) \subset \mathcal{F}(X, Y)$ of all bounded (linear and nonlinear) functions $f : X \rightarrow Y$. Now we have defined the operator norm on $\mathfrak{BL}(X, Y)$ of all linear operators (functions) $T : X \rightarrow Y$. We show in the next example that these two norms are very different.

Example 517 In Example 510 we had $(X, \|\cdot\|_X) = (\mathbb{R}, |\cdot|) = (Y, \|\cdot\|_Y)$ and $T : X \rightarrow Y$ be given by $Tx = 2x$. T is not bounded on \mathbb{R} with respect to the sup norm since $\|2x\|_\infty = \sup\{|2x|, x \in \mathbb{R}\} = \infty$. However the operator norm is bounded in the sense of Definition 509 since $\|2x\| = \inf \{K : K > 0 \text{ and } |2x| \leq K|x|, x \in \mathbb{R}\} = 2$.

In the remainder of this section and the next, when we refer to the norm of a linear operator we mean the norm given in (6.3) if not specified otherwise.

In the following theorem, we show that the norm of a linear operator can be expressed in many different ways.

Theorem 518 The norm of a bounded linear operator $T : X \rightarrow Y$ can be expressed as: (i) $\|T\| = \inf \{K : K > 0 \text{ and } \|Tx\|_Y \leq K \|x\|_X, x \in X\}$; (ii) $\|T\| = \sup \{\|Tx\|_Y, x \in X, \|x\|_X \leq 1\}$; (iii) $\|T\| = \sup \{\|Tx\|_Y, x \in X, \|x\|_X = 1\}$; (iv) $\|T\| = \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X}, x \in X, x \neq 0 \right\}$.

⁵We cannot take $[-3, 3] \subset \mathbb{R}$ since X is supposed to be a vector subspace but $[-3, 3]$ is not because, for instance, it is not closed under scalar multiplication (e.g. if we take the scalar 4 we have $[-12, 12] \not\subseteq [-3, 3]$).

Proof. Denote the right hand sides of expressions (i), (ii), (iii), and (iv) as M_1, M_2, M_3, M_4 . We want to show that $M_1 = M_2 = M_3 = M_4$.

From (i), we have $\|Tx\|_Y \leq M_1 \|x\|_X, \forall x \in X$. Now if $\|x\|_X \leq 1$, then $\|Tx\|_Y \leq M_1$. Since M_2 is the supremum of such a set, then $M_2 \leq M_1$.

Since $\sup \{\|Tx\|_Y, x \in X, \|x\|_X = 1\} \subset \sup \{\|Tx\|_Y, x \in X, \|x\|_X \leq 1\}$, then $M_3 \leq M_2$.

Next, since $\frac{\|Tx\|_Y}{\|x\|_X} = \left\| T \left(\frac{x}{\|x\|_X} \right) \right\|_Y$ for $x \neq 0$, if we let $z = \frac{x}{\|x\|_X}$, then $\|z\|_X = \left\| \frac{x}{\|x\|_X} \right\|_X = \frac{\|x\|_X}{\|x\|_X} = 1$ and hence $M_3 = M_4$.

From the definition of M_4 , it follows that if $\|x\|_X \neq 0$, then $\frac{\|Tx\|_Y}{\|x\|_X} \leq M_4$ or $\|Tx\|_Y \leq M_4 \|x\|_X$. Since M_1 is the infimum, we have $M_1 \leq M_4$.

Thus we have $M_1 \leq M_4 = M_3 \leq M_2 \leq M_1$, which implies the desired result. ■

Corollary 519 *Let X, Y be normed vector spaces and let $T : X \rightarrow Y$ be a bounded linear operator. Then $\|Tx\|_Y \leq \|T\| \cdot \|x\|_X$.*

The next theorem establishes the most important result of this section; namely that $\mathfrak{BL}(X, Y)$ is a complete normed vector space provided that $(Y, \|\cdot\|_Y)$ is complete. We cannot use the previous result on completeness in function spaces (Theorem 449) because $\mathfrak{BL}(X, Y)$ is equipped with a different norm. However, the proof is similar to that used to establish that $\mathfrak{B}(X, Y)$ is complete whenever $(Y, \|\cdot\|_Y)$ is complete.

Theorem 520 *The space $\mathfrak{BL}(X, Y)$ of all bounded linear operators from a normed vector space X to a complete normed vector space Y is itself a complete normed vector space.*

Proof. (Sketch) Let $\langle T_n \rangle$ be a Cauchy sequence in $\mathfrak{BL}(X, Y)$. For fixed $x \in X$, $\langle T_n(x) \rangle$ is Cauchy in Y . Since Y is complete, $\langle T_n(x) \rangle$ converges to an element in Y , call it Tx . Thus we can define an operator $T : X \rightarrow Y$ by $Tx = \lim_{n \rightarrow \infty} T_n(x)$. It is easy to show that T is bounded and that $\langle T_n \rangle \rightarrow T$ in $\mathfrak{BL}(X, Y)$. ■

6.4 Linear Functionals

In this section we study the special case of linear operators that map elements (in this case functions) from a normed vector space to \mathbb{R} .

Definition 521 Let $(X, \|\cdot\|_X)$ be a normed vector space. A linear operator $F : X \rightarrow \mathbb{R}$ is called a **linear functional**. That is, a linear functional is a real-valued function F on X such that $F(\alpha x + \beta x') = \alpha F(x) + \beta F(x')$, $\forall x, x' \in X$ and $\alpha, \beta \in \mathbb{R}$

We note that if X is a finite dimensional vector space (e.g. \mathbb{R}^n), then F is usually called a function. The functional nomenclature is typically used when X is an infinite dimensional vector space (e.g. $\ell_p, \mathcal{C}(X), L_p$).

Definition 522 $F : X \rightarrow \mathbb{R}$ is said to be **bounded** on X if $\exists K \in \mathbb{R}_{++}$ such that $|F(x)| \leq K \cdot \|x\|_X, \forall x \in X$.

Since a bounded linear functional is a special case of a bounded linear operator, everything we proved in the previous Section 6.3 is also valid for linear functionals. We summarize it in the following Theorem.

Theorem 523 Let F be a linear functional on a normed vector space X . Then: (i) F is continuous iff F is continuous at any point in X ; (ii) F is continuous iff F is bounded; (iii) The set of all bounded linear functionals is a complete vector space with the norm of F defined by $\|F\| = \sup \{|F(x)|, x \in X, \|x\|_X \leq 1\}$ or by any other equivalent formula from Theorem 518.

Proof. Follows proofs in the previous section. Part (iii) uses fact that $(\mathbb{R}, |\cdot|)$ is complete (so that the set of all bounded linear functionals is always complete). ■

We note that the set of all bounded linear functionals on X has a special name.

Definition 524 Given a normed vector space X , the set of all bounded linear functionals on X is called the **dual** of X , denoted X^* .

The next set of examples illustrate functionals on finite and infinite dimensional vector spaces.

Example 525 Let \mathbb{R}^n be n -dimensional Euclidean space with the Euclidean norm. Let $a = (a_1, \dots, a_n)$ be a fixed non-zero vector in \mathbb{R}^n . Define the “inner

(or dot) product" functional $F_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ by $F_1 = \langle a, x \rangle = a_1x_1 + \dots + a_nx_n$.⁶ It is clear that F_1 is linear since

$$\begin{aligned} F_1(\alpha x + \beta x') &= \langle a, (\alpha x + \beta x') \rangle \\ &= \alpha \langle a, x \rangle + \beta \langle a, x' \rangle \\ &= \alpha F_1(x) + \beta F_1(x'). \end{aligned}$$

It is also easily established that F_1 is bounded since by the Cauchy-Schwartz inequality we have $|F_1(x)| = |\langle a, x \rangle| \leq \|a\|_X \|x\|_X, \forall x \in \mathbb{R}^n$. Finally, since $\|F_1\| = \sup\{|F_1(x)|, x \in X, \|x\|_X \leq 1\} \leq \|a\|_X$ and $\|F_1\| \geq \frac{|F_1(a)|}{\|a\|_X} = \frac{\|a\|_X^2}{\|a\|_X} = \|a\|_X$, we have $\|F_1\| = \|a\|_X$. Figure 6.4.1 illustrates such functionals in \mathbb{R}^2 .

Example 526 Consider the Banach space $(\ell_1, \|\cdot\|_1)$. Define the linear functional $F_2 : \ell_1 \rightarrow \mathbb{R}$ by $F_2(x) = \sum_{i=1}^{\infty} x_i$ where $x = \langle x_i \rangle_{i=1}^{\infty}$. Then $|F_2(x)| \leq \sum_{i=1}^{\infty} |x_i| = \|x\|_1, \forall x \in \ell_1$. This implies that F_2 is bounded and that $\|F_2\| \leq 1$. Also for $x = e_1 = (1, 0, \dots) \in \ell_1$ we have $\|F_2\| \geq \frac{|F_2(e_1)|}{\|e_1\|_1} = \frac{1}{1} = 1$. Combining these two inequalities yields $\|F_2\| = 1$.

Example 527 Let $X = \mathcal{C}([a, b], \|\cdot\|_{\infty})$. Define the functional $F_3 : X \rightarrow \mathbb{R}$ by $F_3(x) = \int_{[a, b]} x(\omega) d\omega, x \in X$. We can interpret this as the expectation of a random variable drawn from a uniform distribution on support $[a, b]$. It is clear that F_3 is linear. To see that F_3 is bounded, note that

$$\begin{aligned} |F_3(x)| &= \left| \int_{[a, b]} x(\omega) d\omega \right| \leq \int_{[a, b]} |x(\omega)| d\omega \\ &\leq \sup_{\omega \in [a, b]} |x(\omega)| \cdot (b - a) = (b - a) \cdot \|x\|_{\infty}, \forall x \in X. \end{aligned}$$

On the other hand, if $x = x_0$ where $x_0(\omega) = 1 \forall \omega \in [a, b]$, then $\|x_0\|_{\infty} = 1$ and $|F_3(x_0)| = \int_{[a, b]} 1 d\omega = b - a$. Hence $\|F_3\| \geq \frac{|F_3(x_0)|}{\|x_0\|_{\infty}} = b - a$ and combining these inequalities $\|F_3\| = b - a$.

Example 528 Reconsider Example 526 with a different norm. In particular, let $X = (\ell_1, \|\cdot\|_{\infty})$ and let the linear functional $F_4 : \ell_1 \rightarrow \mathbb{R}$ by $F_4(x) = \sum_{i=1}^{\infty} x_i$ where $x = \langle x_i \rangle_{i=1}^{\infty} \in \ell_1$. In this case, F_4 is unbounded. To see this

⁶We introduced this notation in Definition 209.

defined the sequence $\langle x_n \rangle \in \ell_1$ as a sequence of 1's in the first n places and zeros otherwise (i.e. $\langle x_n \rangle = \langle 1, \dots, 1, 0, 0, \dots \rangle$ where the last 1 occurs in the n^{th} place). Then $\|x_n\|_\infty = \sup\{|x_i|, i \in \mathbb{N}\} = 1$ and $F_4(x_n) = n$. Thus, $|F_4(x_n)| = n \cdot \|x_n\|_\infty$, with $\|x_n\|_\infty = 1, \forall n \in \mathbb{N}$. Therefore, $\|F_4\| = \infty$.

6.4.1 Dual spaces

As you may have noticed from Examples 525 to 528, it is quite simple to determine whether “something” is a bounded linear functional. But now we move on to tackle the converse. Given a normed vector space X , is it possible to represent (or characterize) *all* bounded linear functionals on X ? In other words, we want to determine the dual of X . Here we simply consider the dual of some of the most common normed vector spaces.

The dual of the euclidean space \mathbb{R}^n

In Example 525 of this section, we showed that a functional $F_1 : X \rightarrow \mathbb{R}$ defined by $F_1(x) = \langle a, x \rangle$ where $x \in X = \mathbb{R}^n$ and $a \in \mathbb{R}^n$ is a bounded linear functional with $\|F_1\| = \|a\|_X$. The functional F_1 is represented by the point a ; that is, if we vary a , we vary F_1 . Let \mathfrak{F} be the set of all such F_1 . In the case where $X = \mathbb{R}^2$, F_1 are just planes and \mathfrak{F} is the set of all planes. Obviously, $\mathfrak{F} \subset X^*$. Now we show that there are no others. That is, $\mathfrak{F} \supset X^*$ so that $\mathfrak{F} = X^*$.

Theorem 529 *The dual space of \mathbb{R}^n is \mathbb{R}^n itself. That is, each bounded linear functional G on \mathbb{R}^n can be represented by an element $b \in \mathbb{R}^n$ such that $G(x) = \langle b, x \rangle$ for all $x \in \mathbb{R}^n$.*

Proof. (Sketch) Let $G \in (\mathbb{R}^n)^*$ (i.e. G is a bounded linear functional on \mathbb{R}^n). Let $\{e^1, \dots, e^n\}$ be the natural basis in \mathbb{R}^n . Define $b_i = G(e^i)$ for $i = 1, \dots, n$. Then the point $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ represents G . That is, for $x \in \mathbb{R}^n$ we have

$$G(x) = \langle x, b \rangle. \quad (6.4)$$

By the Cauchy-Schwartz inequality we have $\|G\| \leq \|b\|_X$ and by plugging $x = b$ in (6.4) we obtain $\|G\| \geq \|b\|_X$ so that $\|G\| = \|b\|_X$.

This equality establishes that an operator $T : X^* \rightarrow X$ defined by $T(G) : (G(e^1), \dots, G(e^n)) = b$ is an *isometry* (see Definition 171). This means that

T preserves distances and hence it preserves topological properties of spaces $(X, \|\cdot\|_X)$ and $(X^*, \|\cdot\|)$.

It is easy to verify that T is a bijection and that T is a linear operator. Hence T preserves the algebraic (in this case linear) structure of these two spaces. In this case we say that T is an *isomorphism*.

Putting these two together we have that T is an isometric isomorphism between $(X, \|\cdot\|_X)$ and $(X^*, \|\cdot\|)$ and hence these two spaces are indistinguishable from the point of view of the number of elements, as well as the algebraic and topological structure. Hence they are effectively the same space with differently named elements. ■

The dual of a separable hilbert space

Since Euclidean space is a separable, complete inner product space, one might like to know if there is a similar result to Theorem 529 for any separable Hilbert space. The answer is yes.

Theorem 530 *The dual of a separable Hilbert space \mathcal{H} is \mathcal{H} itself. That is, for every bounded linear functional F on a separable, complete inner product space \mathcal{H} , there is a unique element $y \in \mathcal{H}$ such that: (i) $F(x) = \langle x, y \rangle$, $\forall x \in \mathcal{H}$; and (ii) $\|F\| = \|y\|$.*

Proof. (Sketch) By Theorem 501a separable Hilbert space contains a countable, complete orthonormal basis $\{e^i, i \in \mathbb{N}\}$. Let F be a bounded linear functional on \mathcal{H} . Set $b_i = F(e^i)$, $i = 1, 2, \dots$. It is easy to show that $\sum_{i=1}^{\infty} b_i^2 \leq \|F\|^2 < \infty$. Hence by Parseval's Theorem 504 there exists a $b \in \mathcal{H}$ such that $b = \sum_{i=1}^{\infty} b_i e^i$ where b_i are the Fourier coefficients of b . Moreover, $\|b\|_{\mathcal{H}} \leq \|F\|$. Let $x = \sum_{i=1}^{\infty} x_i e^i$ where x_i are the Fourier coefficients of x . Then by Parseval's equality $x_n (= \sum_{i=1}^n x_i e^i) \rightarrow x (= \sum_{i=1}^{\infty} x_i e^i)$ as $n \rightarrow \infty$. Furthermore because F is continuous and linear

$$F(x) = \lim_{n \rightarrow \infty} F\left(\sum_{i=1}^n x_i e^i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i F(e^i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i b_i = \sum_{i=1}^{\infty} x_i b_i = \langle x, b \rangle .$$

so that $\|F\| \leq \|b\|_{\mathcal{H}}$. ■

A similar result can be proven for a nonseparable, complete inner product space. Thus we can conclude that the dual space of any Hilbert space is a Hilbert space itself (i.e. $\mathcal{H}^* = \mathcal{H}$).

Since ℓ_2 and $L_2([a, b])$ are separable Hilbert spaces, we have $\ell_2^* = \ell_2$ and $L_2^*([a, b]) = L_2([a, b])$. In the case of $L_2([a, b])$, we can claim that for any bounded, linear functional $F : L_2([a, b]) \rightarrow \mathbb{R}$ there exists a *unique* function $g \in L_2([a, b])$ such that $F(f) = \int_a^b g f dx, \forall f \in L_2([a, b])$. This will be shown in Theorem 532.

The dual space of ℓ_p

While the previous section applied to inner product spaces, what about the dual space to a complete normed vector space that is not a Hilbert space? In this section, we consider the dual of ℓ_p for $p \neq 2$.

Let $p \in [1, \infty)$ and let $z \in \ell_q$ where p, q are conjugate. Then $F : \ell_p \rightarrow \mathbb{R}$ given by

$$F(x) = \sum_{i=1}^{\infty} x_i z_i \text{ for } x = \langle x_i \rangle_{i=1}^{\infty} \in \ell_p \quad (6.5)$$

is a bounded linear functional on ℓ_p . This follows immediately from Holder's inequality (Theorem 479).

We now show that all bounded linear functionals on ℓ_p are of the form (6.5).

Theorem 531 *Let $p \in [1, \infty)$ and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $F \in \ell_p^*$, there exists an element $z = \langle z_i \rangle \in \ell_q$ such that*

$$F(x) = \sum_{i=1}^{\infty} x_i z_i$$

for all $x = \langle x_i \rangle_{i=1}^{\infty} \in \ell_p$ and $\|F\| = \|z\|_q$.

Proof. (Sketch) Let F be a bounded linear functional on ℓ_p . Let $\{e^i, i \in \mathbb{N}\}$ be the set of vectors having the i -th entry equal to one and all other entries equal to zero. Set $F(e^i) = z_i, i \in \mathbb{N}$. Given $x = \langle x_1, x_2, \dots \rangle \in \ell_p$, let s_n be the vector consisting of the first n coordinates of x (i.e. $s_n = \sum_{i=1}^n x_i e^i$). Then $s_n \in \ell_p$ and $\|x - s_n\|_p^p = \sum_{i=n+1}^{\infty} |x_i|^p \rightarrow 0$ as $n \rightarrow \infty$. Due to linearity and continuity of F , $F(x) = \sum_{i=1}^{\infty} x_i F(e^i) = \sum_{i=1}^{\infty} x_i z_i$ and $\|F\| \leq \|z\|_q$. By plugging $x = \langle x_i \rangle$ where

$$x_i = \begin{cases} |z_i|^{q-2} z_i & \text{when } z_i \neq 0 \\ 0 & \text{when } z_i = 0 \end{cases}$$

we get that $\|z\|_q \leq \|F\|$. This shows that $\|F\| = \|z\|_q$. ■

Theorem 531 establishes that $\ell_p^* = \ell_q$ where p and q are conjugate. Thus, the dual space of ℓ_1 is ℓ_∞ . However, the reverse is not true. That is, the dual space of ℓ_∞ is not ℓ_1 or $\ell_\infty^* \not\supseteq \ell_1$. We show this in the next section.

The dual space of L_p

An important theorem, known as the Riesz representation theorem, establishes a result similar to Theorem 531 for L_p . Let $X \subset \mathbb{R}$, $1 \leq p < \infty$ and q conjugate to p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). Let $g \in L_q(X)$. Define a functional $F : L_p \rightarrow \mathbb{R}$ by

$$F(f) = \int_X f g dm$$

for all $f \in L_p(X)$. It is easy to see that F is a bounded linear functional on $L_p(X)$. Linearity follows from linearity of the integral and boundedness follows from the Holder inequality. Then we have the result that each linear functional on $L_p(X)$ can be obtained in this manner (i.e. $L_p^* = L_q$).

Theorem 532 (Riesz Representation) *Let F be a bounded linear functional on $L_p(X)$ and $1 \leq p < \infty$. Then there is a function $g \in L_q(X)$ such that*

$$F(f) = \int_X f g dm$$

and $\|F\| = \|g\|_q$.

Proof. (Sketch) Let F be a bounded linear functional on L_p . In all the previous cases, in finding the element b that represents a given functional we used the same procedure; we set $b_i = F(e^i)$ where $\{e^i\}$ is a basis. In L_p , we use indicator functions.

First assume that $m(X) < \infty$ (later we relax this assumption). For any $E \subset X$ which is \mathcal{L} -measurable (i.e. $E \in \mathcal{L}$), $\chi_E \in L_p(X)$. Thus given F we define a set function $\nu : \mathcal{L} \rightarrow \mathbb{R}$ by $\nu(E) = F(\chi_E)$ for $E \subset \mathcal{L}$. ν is a finite signed measure which is absolutely continuous with respect to m . Then by the Radon Nikodym Theorem 434 there is an \mathcal{L} -integrable function g that represents ν (i.e. $\nu(E) = \int_E g dm = \int_X \chi_E g dm$). By linearity of F we have

$$F(\varphi) = \int_X g \varphi dm$$

for all simple functions $\varphi \in L_p(X)$ and $|F(\varphi)| \leq \|F\| \|\varphi\|_p$. Then it can be shown that $g \in L_q(X)$. Because the set of simple functions is dense in $L_p(X)$, then $F(f) = \int_X g f dm$ for all $f \in L_p(X)$.

If $m(X) = \infty$, since m is σ -finite, there is an increasing sequence of \mathcal{L} -measurable sets $\langle X_n \rangle$ with finite measure whose union is X . Thus we apply the result proven in the first part of the proof to define $\langle g_n \rangle$ on X_n . Then show that $g_n \rightarrow g$ and $F(f) = \int f g_n dm$ for all $f \in L_p$. ■

Here we note that the dual of $L_\infty(X)$ is not $L_1(X)$. That is, not all bounded functionals on $L_\infty([a, b])$ can be represented by $F(f) = \int_X f g$, where $g \in L_1(X)$. The proof of this result is easier to see after some future results on separation, so we wait until then.

6.4.2 Second Dual Space

In the previous section we showed that the dual space X^* of all bounded linear functionals defined on a normed linear space X is a normed linear space itself. Then it is possible to speak of the space $(X^*)^*$ of bounded linear functionals defined on X^* which is called the *second dual space* X^{**} of X . Of course X^{**} is also a normed vector space.

Let us try to define some elements of X^{**} . Given a fixed element x_0 in X we can define a functional $\psi : X^* \rightarrow \mathbb{R}$ by $\psi_{x_0}(f) = f(x_0)$ where f runs through all of X^* . Notice that ψ assigns to each element $f \in X^*$ its value at a certain fixed element of X . We have $\psi_{x_0}(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x_0) = \alpha f_1(x_0) + \beta f_2(x_0) = \alpha \psi_{x_0}(f_1) + \beta \psi_{x_0}(f_2)$ (since f_1, f_2 are linear functionals) and $|\psi_{x_0}(f)| = |f(x_0)| \leq \|f\| \|x_0\|$ (since f is bounded). Hence ψ_{x_0} is a bounded linear functional on X^* .

Besides the notation $f(x)$, which indicates the value of the functional f at a point x , we will find it useful to employ the symmetric notation $f(x) \equiv \langle f, x \rangle$. It is not a coincidence that a value of a functional is denoted the same way as the scalar product because any bounded linear functional g defined on a Hilbert space can be represented by a scalar product (i.e. $\exists y \in \mathcal{H}$ such that $g(x) = \langle y, x \rangle$, $\forall x \in \mathcal{H}$ by Theorem 530).

For fixed $f \in X^*$ we can consider $\langle f, x \rangle$ as a functional on X and for fixed $x \in X$ as a functional on X^* (i.e. as an element of X^{**}). Let us define a new norm $\|\cdot\|_2$ on X by the following

$$\|x\|_2 = \sup \left\{ \frac{|\langle f, x \rangle|}{\|f\|}, f \in X^*, f \neq 0 \right\}$$

How is this norm related to the original norm $\|\cdot\|_X$ on X ? We shall show that $\|x\|_X = \|x\|_2$. Let f be an arbitrary non zero element in X^* . Then

$$|\langle f, x \rangle| \leq \|f\| \|x\|_X \Leftrightarrow \|x\|_X \geq \frac{|\langle f, x \rangle|}{\|f\|}.$$

Since this inequality is true for any f then

$$\|x\|_X \geq \sup \left\{ \frac{|\langle f, x \rangle|}{\|f\|}, f \in X^*, f \neq 0 \right\} = \|x\|_2. \quad (6.6)$$

Now to the converse inequality. By Theorem 540 of the next section, for any element $x \in X$, $x \neq 0$ there is a bounded linear functional f_0 such that

$$|\langle f_0, x \rangle| = \|f_0\| \|x\|_X \Leftrightarrow \frac{|\langle f_0, x \rangle|}{\|f_0\|} = \|x\|_X.$$

Consequently

$$\|x\|_2 = \sup \left\{ \frac{|\langle f, x \rangle|}{\|f\|}, f \in X^*, f \neq 0 \right\} \geq \|x\|_X \quad (6.7)$$

Combining inequalities (6.6) and (6.7), we have that $\|x\|_2 = \|x\|_X$.

Since $\langle f, x \rangle$ for fixed $x \in X$ is a linear functional on X^* , then by (iv) of Theorem 518, the expression $\sup \left\{ \frac{|\langle f, x \rangle|}{\|f\|}, f \in X^*, f \neq 0 \right\}$ is the norm of this functional. But this expression is identical to $\|\cdot\|_2$. If we now define a mapping $J : X \rightarrow X^{**}$ by $J(x) = \langle f, x \rangle, f \in X^*$, then by the virtue of the identity $\|x\|_X = \|x\|_2 = \|J(x)\|$, the space X is isometric with some subset F of X^{**} . See Figure 6.4.2.1. Thus X and $F \subset X^{**}$ are isometrically isomorphic (i.e. they are indistinguishable so we may write $X = F$ and $X \subset X^{**}$).

There is a class of normed vector spaces X for which the mapping $J : X \rightarrow X^*$ is onto (i.e. $X = X^*$).

Definition 533 *The space X is said to be **reflexive** if $X = X^{**}$.*

As we will see later this property plays a very important role in optimization theory. Let us check some known vector spaces for reflexivity.

Example 534 *The Euclidean space \mathbb{R}^n is reflexive. Why? We showed in the previous section (Theorem 529) that even the first dual of \mathbb{R}^n is \mathbb{R}^n (i.e. $(\mathbb{R}^n)^* = \mathbb{R}^n$). Hence $(\mathbb{R}^n)^{**} = ((\mathbb{R}^n)^*)^* = (\mathbb{R}^n)^* = \mathbb{R}^n$.*

Example 535 Infinite dimensional Hilbert spaces (ℓ_2, L_2) are reflexive. By Theorem 530 we have that $\mathcal{H}^* = \mathcal{H}$ which means that any Hilbert space is reflexive. That is, $\ell_2^{**} = \ell_2$, $L_2^{**} = L_2$.

What about ℓ_p, L_p when $p \neq 2$?

Example 536 If $1 < p < \infty$, then by Theorem 531, $\ell_p^* = \ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Thus $\ell_p^{**} = (\ell_p^*)^* = \ell_q^* = \ell_p$ because p, q are mutually conjugate. Similarly $L_p^{**} = L_p$.

Thus if $1 < p < \infty$, then ℓ_p, L_p are reflexive. If $p = 1$ by Theorem 531 $\ell_1^* = \ell_\infty$. But $\ell_\infty^* \supsetneq \ell_1$. Hence $(\ell_1^*)^* = \ell_\infty^* \supsetneq \ell_1$ so that ℓ_1 is not reflexive. ℓ_∞ is also not reflexive. Similarly L_1, L_∞ are not reflexive. We will show this in the next section.

Example 537 It can be shown that the space $C([a, b])$ of all continuous functions on $[a, b]$ is not reflexive.

Exercise 6.4.1 Show that X is reflexive iff X^* is reflexive.

6.5 Separation Results

In this section we state and prove probably the most important theorem in functional analysis; the Hahn-Banach theorem. It has numerous applications. We will concentrate on a geometric application and will formulate it as a separation result for convex sets. Using this theorem we prove the existence of a competitive equilibrium allocation in a general setting.

First we define a new notion.

Definition 538 Let X be a normed vector space. A functional $P : X \rightarrow \mathbb{R}$ is called **sublinear** if: (i) $P(x + x') \leq P(x) + P(x')$, $\forall x, x' \in X$; and (ii) $P(\alpha x) = \alpha P(x)$, $\forall x \in X$ and $\alpha \in \mathbb{R}_{++}$.

Exercise 6.5.1 Let X be a normed vector space. Show that the norm $\|\cdot\|_X : X \rightarrow \mathbb{R}$ is a sublinear functional.

The Hahn-Banach theorem provides a method of constructing bounded linear functionals on X with certain properties. One first defines a bounded linear functional on a subspace of a normed vector space where it is easy to verify the desired properties. Then the theorem guarantees that this functional can be extended to the whole space while retaining the desired properties.

Theorem 539 (Hahn-Banach) *Let X be a vector space and $P : X \rightarrow \mathbb{R}$ be a sublinear functional on X . Let M be a subspace of X and let $f : X \rightarrow \mathbb{R}$ be a linear functional on M satisfying*

$$f(x) \leq P(x), \forall x \in M. \quad (6.8)$$

Then there exists a linear functional $F : X \rightarrow \mathbb{R}$ on the whole of X such that

$$F(x) = f(x), \forall x \in M \quad (6.9)$$

and

$$F(x) \leq P(x), \forall x \in X. \quad (6.10)$$

Proof. (Sketch) Choose $x_1 \in X \setminus M$. Define a linear subspace $M_1 = \{x : x = \alpha x_1 + y, y \in M\}$. Let F be an extension of f to M_1 . Since F is linear, then $F(\alpha x_1 + y) = \alpha F(x_1) + F(y) = \alpha F(x_1) + f(y)$. Thus F is completely determined by $F(x_1)$.

Next we derive lower and upper bounds for $F(x_1)$ in order for F to satisfy (6.9) and (6.10) for $x \in M_1$. Thus F is an extension of f from M to M_1 where $M \subsetneq M_1$. This process can be repeated and Zorn's lemma guarantees that F can be extended to the whole space X .

In order to apply Zorn's lemma we define a partial order on the set $S = \{\text{all linear functionals } g : D \rightarrow \mathbb{R} \text{ where } D \text{ is a subspace of } X \text{ and } g(x) \leq P(x), \forall x \in D\}$ in the following way. Let g_1, g_2 . Then $g_1 < g_2$ if $D(g_1) \subset D(g_2)$ and $g_1(x) = g_2(x), \forall x \in D(g_1)$. Then we must check that every totally ordered subset of S has an upper bound (in which case the assumptions of Zorn's lemma are satisfied). ■

At first sight the Hahn-Banach Theorem 539 doesn't look like a "big deal". Its significance in functional analysis, however, becomes apparent through its wide range of applications, many of them involving a clever choice of the subadditive functional P . We will state just three propositions.

The first result says that a bounded linear functional defined on a vector subspace can be extended on the whole vector space.

Theorem 540 *Let M be a vector subspace of a normed vector space X . Let f be a bounded linear functional on M , then there exists a bounded linear functional F on X s.t. $F(x) = f(x)$, $\forall x \in M$ and $\|F\| = \|f\|$.*

Proof. The function $P(x) = \|f\| \|x\|$ is a sublinear functional on X and $|f(x)| \leq \|f\| \|x\| = p(x)$ (See Exercise 6.5.1). Then by Hahn-Banach Theorem, $\exists F$ (an extension of f on X) with property that $|F(x)| \leq p(x) = \|f\| \|x\|$, $\forall x \in X$. This means that F is bounded on X and also $\|F\| \leq \|f\|$. Because F is an extension of f , then $\|f\| \leq \|F\|$. Hence $\|F\| = \|f\|$. ■

Exercise 6.5.2 *Carefully compare the assumptions of the Hahn-Banach Theorem 539 and Theorem 540.*

Theorem 540 can be used to show that the dual of $L_\infty([a, b])$ is not $L_1([a, b])$.

Lemma 541 *Not all bounded functionals on $L_\infty([a, b])$ can be represented by $F(f) = \int_{[a,b]} fg$, where $g \in L_1([a, b])$. That is, $(L_\infty([a, b]))^* \not\supseteq L_1([a, b])$.*

Proof. $C([a, b])$ is a vector subspace of $L_\infty([a, b])$. Let $F_1 : C([a, b]) \rightarrow \mathbb{R}$ be a linear functional which assigns to each $f \in C([a, b])$ the value $f(a)$ (i.e. $F_1(f) = f(a)$). Since $\|F_1\| = \sup \left\{ \frac{|F_1(f)|}{\|f\|_{C([a,b])}}, \|f\| \neq 0 \right\} = \sup \left\{ \frac{|f(a)|}{\sup\{|f(x)|, x \in [a,b]\}} \right\} \leq 1$, F_1 is bounded and by Theorem 540 F_1 can be extended to a bounded linear functional F on the whole set $L_\infty([a, b])$. Let's assume, by contradiction, that there is $g \in L_1([a, b])$ such that F can be represented by $F(f) = \int_a^b fgd x$, $\forall f \in C([a, b])$. Let $\langle h_n \rangle$ be a sequence of continuous functions on $[a, b]$ which are bounded by 1, have $h_n(a) = 1$, and such that $h_n(x) \rightarrow 0$ for all $x \neq a$. For example set $h_n(x) = \left[\frac{1}{b-a} (b-x) \right]^n$. Then for each $g \in L_1$, $\int_a^b h_n g \rightarrow 0$ by the Bounded Convergence Theorem 386). Since $F(h_n) = \int_a^b g h_n$ by assumption, we have $F(h_n) \rightarrow 0$. But $F(h_n) = h_n(a) = 1$ for all n , which is a contradiction. ■

Corollary 542 $L_1(X)$ and $L_\infty(X)$ are not reflexive.

Proof. We know by Theorem 532 that $L_1^* = L_\infty$ and by Lemma 541 that $L_\infty^* \not\supseteq L_1$. Combining these two results we have the $(L_1^*)^* = L_\infty^* \not\supseteq L_1$. Furthermore since L_1 is not reflexive, neither is L_∞ by Exercise 6.4.1. ■

The second result states that given a normed vector space X , its dual X^* has “sufficiently” many elements (i.e. at least as many elements as X itself).

Theorem 543 *Let X be a normed vector space and let $x_0 \neq 0$ be any element of X . Then there exists a bounded linear functional F on X such that $\|F\| = 1$ and $F(x_0) = \|x_0\|$.*

Proof. Let M be the subspace consisting of all multiples of x_0 (i.e. $M = \{\alpha x_0, \alpha \in \mathbb{R}\}$). Define $f : M \rightarrow \mathbb{R}$ by $f(\alpha x_0) = \alpha \|x_0\|$. Then f is a linear functional on M . Define $P : X \rightarrow \mathbb{R}$ by $P(y) = \|y\|$. P is a sublinear functional on X satisfying $f(x) \leq P(x)$ for $x \in M$. Then by the Hahn-Banach theorem there exists a linear functional $F : X \rightarrow \mathbb{R}$ that is an extension of f and $F(x) \leq P(x) = \|x\|, \forall x \in X$. For $-x$, we have $F(-x) \leq \| -x \| = \|x\|$ in which case $|F(x)| \leq \|x\|, \forall x \in X$. Thus F is bounded and $\|F\| = \sup \left\{ \frac{|F(x)|}{\|x\|}, \forall x \in X, x \neq 0 \right\} = 1$. Also, since F is an extension of f , $F(x_0) = F(1 \cdot x_0) = 1 \cdot \|x_0\| = \|x_0\|$. ■

The third proposition is a geometric version of the Hahn-Banach theorem. It is a separation result for convex sets. Before stating it we have to introduce a few geometric concepts.

Definition 544 *Let $K \subset X$ be convex. A point $x \in K$ is an **internal point** of a convex set K if given any $y \in X, \exists \varepsilon > 0$ such that $x + \delta y \in K$ for all δ satisfying $|\delta| < \varepsilon$.*

Geometrically, the statement that x is an internal point of K means that the intersection of K with any line L through x contains a segment with x as a midpoint. See Figure 6.5.1.

Definition 545 *Let $\mathbf{0}$ (a zero vector) be an internal point of a convex set K . Then the **support function** $P : X \rightarrow \mathbb{R}_{++}$ of K (with respect to $\mathbf{0}$) is given by*

$$P(x) = \inf \left\{ \lambda : \frac{x}{\lambda} \in K, \lambda > 0 \right\}.$$

The support function has a simple geometric interpretation. Let $x \in X$. Draw the line segment (a ray) from $\mathbf{0}$ through x . There is a point y on this segment that is a boundary point of K . Then the scalar λ for which $\lambda y = x$ is $P(x)$ so that $P(x)y = x$. See Figure 6.5.2.

We have the following properties for this support function.

Lemma 546 *If K is a convex set containing $\mathbf{0}$ as an internal point then the support function P has the following properties: (i) $P(\alpha x) = \alpha P(x)$ for $\alpha \geq 0$; (ii) $P(x + y) \leq P(x) + P(y)$; (iii) $\{x : P(x) < 1\} \subset K \subset \{x : P(x) \leq 1\}$*

Proof. (i) Let $\alpha > 0$,

$$\begin{aligned} P(\alpha x) &= \inf \left\{ \lambda : \frac{\alpha x}{\lambda} \in K, \lambda > 0 \right\} \\ &= \inf \left\{ \alpha \frac{\lambda}{\alpha} : \frac{x}{\frac{\lambda}{\alpha}} \in K, \frac{\lambda}{\alpha} > 0 \right\} \\ &= \inf \left\{ \alpha \beta : \frac{x}{\beta} \in K, \beta > 0 \right\} \\ &= \alpha \inf \left\{ \beta : \frac{x}{\beta} \in K, \beta > 0 \right\} = \alpha P(x) \end{aligned}$$

where $\beta = \frac{\lambda}{\alpha} > 0$.

(ii) Let $\alpha = \inf \left\{ \lambda : \frac{x}{\lambda} \in K, \lambda > 0 \right\} = P(x)$ and $\beta = \inf \left\{ \mu : \frac{y}{\mu} \in K, \mu > 0 \right\} = P(y)$. Take λ and μ such that $\frac{x}{\lambda} \in K$ and $\frac{y}{\mu} \in K$. Then $\alpha \leq \lambda$ and $\beta \leq \mu$. Since K is convex,

$$\left(\frac{\lambda}{\lambda + \mu} \right) \left(\frac{x}{\lambda} \right) + \left(\frac{\mu}{\lambda + \mu} \right) \left(\frac{y}{\mu} \right) \in K$$

because $\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} = 1$. Then $\frac{x+y}{\lambda + \mu} \in K$. Thus $P(x+y) \leq \lambda + \mu$ (because $P(x+y)$ is the infimum of such scalars). Hence

$$P(x+y) \leq \lambda + \mu \leq \alpha + \beta = P(x) + P(y).$$

(iii) It follows from the definition of P . ■

Example 547 Let $X = \mathbb{R}^2$ with the Euclidean norm. Let $K = \{(x_1, x_2) \in \mathbb{R}^2 : \|(x_1, x_2)\| \leq 1\}$. Obviously K (the unit ball) is convex. Consider a point $x^1 = (2, 2)$ outside the ball. Then $P((2, 2)) = \left\{ \lambda : \left(\frac{2}{\lambda}, \frac{2}{\lambda} \right) \in K, \lambda > 0 \right\}$. But $\left(\frac{2}{\lambda}, \frac{2}{\lambda} \right) \in K$ iff $\left\| \left(\frac{2}{\lambda}, \frac{2}{\lambda} \right) \right\| \leq 1 \iff \frac{4}{\lambda^2} + \frac{4}{\lambda^2} \leq 1$ or $\lambda \geq \sqrt{8}$, so $P((2, 2)) = \sqrt{8} > 1$. Now consider a point $x^2 = \left(\frac{1}{2}, \frac{1}{2} \right)$ inside the ball. Then $P\left(\left(\frac{1}{2}, \frac{1}{2}\right)\right) = \sqrt{\frac{1}{2}} < 1$. See Figure 6.5.3.

Definition 548 Two convex sets K_1, K_2 are **separated** by a linear functional F if $\exists \alpha \in \mathbb{R}$ such that $F(x) \leq \alpha, \forall x \in K_1$ and $F(x) \geq \alpha, \forall x \in K_2$.

Theorem 549 (Separation) *Let K_1, K_2 be two convex sets of a normed vector space X . Assume that K_1 has at least one internal point and that K_2 contains no internal point of K_1 . Then there is a nontrivial linear functional separating K_1 and K_2 .*

Proof. (Sketch) Let K_1 and K_2 be two convex subsets of X and without loss of generality let $\mathbf{0} \in K_1$ and $x_0 \in K_2$. Define $K = x_0 + K_1 - K_2$. See Figure 6.5.4. $\mathbf{0}$ is an internal point of K and x_0 is not an internal point of K (this latter fact follows since K_2 contains no internal points of K_1). Thus by (iii) of Lemma 546 $P(x) \leq 1$ for all $x \in K$ and $P(x_0) \geq 1$ where P is a support function of K .

Let M be a vector subspace (i.e. $M = \{x : x = \alpha x_0, \alpha \in \mathbb{R}\}$). Define $f : M \rightarrow \mathbb{R}$ by $f(x) = P(\alpha x_0) = \alpha P(x_0)$. f is a linear functional that satisfies $f(x) \leq P(x) \forall x \in M$. Hence by the Hahn-Banach Theorem 539 there exists an extension of f (i.e. a linear functional $F : X \rightarrow \mathbb{R}$ satisfying $F(x) \leq P(x) \forall x \in X$). This functional F separates K_1 and K_2 . Why? Take $x \in K$ with $x = x_0 + y - z$ where $y \in K_1$ and $z \in K_2$. Then $F(x) \leq P(x) \leq 1$ for $x \in K$. Since F is linear

$$F(x_0) + F(y) - F(z) \leq 1 \iff F(y) + (F(x_0) - 1) \leq F(z) \quad (6.11)$$

Since $x_0 \in M$,

$$F(x_0) = f(x_0) = P(x_0) \geq 1 \iff F(x_0) - 1 \geq 0 \quad (6.12)$$

Combining (6.11) and (6.12) we have $F(y) \leq F(z)$ for any $y \in K_1$ and $z \in K_2$. Hence

$$\sup_{y \in K_1} F(y) \leq \inf_{z \in K_2} F(z).$$

Thus F separates K_1, K_2 and F is a non-zero functional (since $F(x_0) = 1$).

■

There are several corollaries and modifications of this important separation theorem.

Corollary 550 (Separation of a point from a closed set) *If K is a nonempty, closed, convex set and $x_0 \notin K$, then there exists a continuous linear functional F not identically zero such that $F(x_0) < \inf_{x \in K} F(x)$.*

Proof. By translating by $-x_0$, we reduce the Corollary to the case where $x_0 = 0$. Since $x_0 \notin K$ and K is closed, then by Exercise 4.1.3 we have $0 < d = \inf_{x \in K} \|x_0 - x\|$. Let $B_{\frac{d}{2}}(x_0)$ be the open ball around x_0 with radius $\frac{d}{2}$. By the Separation Theorem 549 there exists a linear functional f such that

$$\sup_{x \in B_{\frac{d}{2}}(0)} f(x) \leq \inf_{y \in K} f(y) = \alpha.$$

Thus $f(x) \leq \alpha$ for $x \in B_{\frac{d}{2}}(x_0)$. If $x \in B_{\frac{d}{2}}(x_0)$, then $-x \in B_{\frac{d}{2}}(x_0)$ which implies that $f(-x) = -f(x) \leq \alpha$. Hence $|f(x)| \leq \alpha$ for all $x \in B_{\frac{d}{2}}(x_0)$. This implies continuity at 0 and by Theorem 508 continuity everywhere.

To show strict inequality, take $x \in B_{\frac{d}{2}}(x_0)$ and $\lambda > 0$ such that $\lambda x \in B_{\frac{d}{2}}(x_0)$ (this is possible since 0 is an internal point of $B_{\frac{d}{2}}(x_0)$). We have $0 < \lambda f(x) = f(\lambda x) \leq \alpha$. Thus we have $f(0) = 0 < \alpha = \inf_{y \in K} f(y)$. See Figure 6.5.5. ■

Corollary 551 (Strict Separation) *Suppose that a nonempty, closed, convex set K_1 and a nonempty, compact convex set K_2 are disjoint. Then there exists a continuous linear functional F , not identically zero, that strictly separates them (i.e. $\sup_{x \in K_1} F(x) < \inf_{x \in K_2} F(x)$).*

Proof. If K_1, K_2 are convex, then $K_1 - K_2$ is convex. Since K_1 is closed and K_2 is compact, then $K_1 - K_2$ is closed. Since $K_1 \cap K_2 = \emptyset$, then $0 \notin K_1 - K_2$. Now apply Corollary 550 with $x_0 = 0$ and $K = K_1 - K_2$. See Figure 6.5.6. ■

It doesn't suffice to assume both sets K_1 and K_2 are closed. One of them has to be compact. For an example of this, see Aliprantis and Border Example 5.51. This doesn't contradict the Separation Theorem as it might seem because Theorem 549 requires the additional assumption of the existence of an internal point of at least one of the sets.

Exercise 6.5.3 *Show that if K_1 is closed and K_2 is compact, then $K_1 - K_2$ is closed.*

6.5.1 Existence of equilibrium

Let S be a finite dimensional Euclidean space with norm $\|\cdot\| = (\sum_{i=1}^n |x_i^2|)^{\frac{1}{2}}$. There are I consumers, indexed by $i = 1, \dots, I$. Consumer i chooses among

commodity points in a set $X_i \subset S$ and maximizes utility given by $u_i : X_i \rightarrow \mathbb{R}$. There are J firms, indexed by $j = 1, \dots, J$. Firm j chooses among points in a set $Y_j \subset S$ describing its technological possibilities and maximizes profits.

We say that an $(I + J)$ -tuple $(\{x_i\}_{i=1}^I, \{y_j\}_{j=1}^J)$ describing the consumption x_i of each consumer and the production y_j of each producer is an **allocation** for this economy. An allocation is **feasible** if: $x_i \in X_i, \forall i; y_j \in Y_j, \forall j$; and $\sum_{i=1}^I x_i - \sum_{j=1}^J y_j \leq 0$ (where there is free disposal). An allocation is **Pareto Optimal** if it is feasible and if there is no other feasible allocation $(\{x'_i\}_{i=1}^I, \{y'_j\}_{j=1}^J)$ such that $u_i(x'_i) \geq u_i(x_i), \forall i$ and $u_i(x'_i) > u_i(x_i)$ for some i . An allocation $(\{x_i^*\}_{i=1}^I, \{y_j^*\}_{j=1}^J)$ together with a continuous linear functional $\phi : S \rightarrow \mathbb{R}$ is a **competitive equilibrium** if: (a) $(\{x_i^*\}_{i=1}^I, \{y_j^*\}_{j=1}^J)$ is feasible; (b) for each $i, x \in X_i$, and $\phi(x) \leq \phi(x_i^*)$ implies $u_i(x) \leq u_i(x_i^*)$; and (c) for each $j, y \in Y_j$ implies $\phi(y) \leq \phi(y_j^*)$.

Theorem 552 (Second Welfare Theorem) *Let: (A1) X_i is convex for each i ; (A2) if $x, x' \in X_i, u_i(x) > u_i(x')$ and $\alpha \in (0, 1)$, then $u_i(\alpha x + (1 - \alpha)x') > u_i(x')$ for each i ; (A3) $u_i : X_i \rightarrow \mathbb{R}$ is continuous for each i ; (A4) the set $Y = \sum_{j=1}^J Y_j$ is convex.⁷ Under (A1) – (A4), let $(\{x_i^*\}_{i=1}^I, \{y_j^*\}_{j=1}^J)$ be a Pareto Optimal allocation. Assume that for some $h \in \{1, \dots, I\}, \exists \hat{x}_h$ such that $u_h(\hat{x}_h) > u_h(x_h^*)$. Then there exists a continuous linear functional $\phi : S \rightarrow \mathbb{R}$, not identically zero on S , such that:*

$$\forall i, x \in X_i \text{ and } u_i(x_i) \geq u_i(x_i^*) \Rightarrow \phi(x) \geq \phi(x_i^*) \quad (6.13)$$

and

$$\forall j, y \in Y_j \Rightarrow \phi(y) \leq \phi(y_j^*). \quad (6.14)$$

If

$$\forall i, \exists x'_i \text{ such that } \phi(x'_i) < \phi(x_i^*), \quad (6.15)$$

then $\{(\{x_i^*\}_{i=1}^I, \{y_j^*\}_{j=1}^J), \phi\}$ is a competitive equilibrium.

Proof. (Sketch) Since S is finite dimensional and the aggregate technological possibilities set is convex (A4), for the existence of ϕ it is sufficient to show that the set of allocations preferred to $\{x_i^*\}_{i=1}^I$ given by $A = \sum_{i=1}^I A_i$ is convex where $A_i = \{x \in X_i : u_i(x) \geq u_i(x_i^*)\}, \forall i$ and that A does not contain any interior points of Y . Then apply Theorem 549. To complete the proof,

⁷The assumption that S is finite dimensional is also important, but can be weakened in the infinite dimensional case to assume that Y has an interior point.

it is sufficient to show (b) holds in the definition of a competitive equilibrium which follows from contraposition of (6.13). ■

You should recognize that $\phi(x) = \langle p, x \rangle$ can be considered an inner product representation of prices.

6.6 Optimization of Nonlinear Operators

In this chapter we have dealt with linear operators and functionals. While we showed very deep results in linear functional analysis - the Riesz Representation Theorem and the Hahn Banach Theorem to name just a few - there are many problems in economics that involve nonlinear operators. For instance, the operator in most dynamic programming problems, such as the growth example suggested in the introduction to this chapter, does not satisfy the linearity property of an operator. In particular, an operator $T : X \rightarrow Y$ as simple as $T(x) = a + bx$ does not possess the linearity property since $T(\alpha x + \beta x') = a + b(\alpha x + \beta x') \neq \alpha T x + \beta T x'$. Such a function does possess a monotonicity property (i.e. if $x \leq x'$, then $T x \leq T x'$).

Nonlinear functional analysis is a very broad area covering topics such as fixed points of nonlinear operators (which we touched on a subsection of 6.1), nonlinear monotone operators, variational methods and optimization of nonlinear operators. In this section, we show how variational methods and fixed point theory (in the form of dynamic programming) can be used to prove the existence of an optimum of a nonlinear operator.

6.6.1 Variational methods on infinite dimensional vector spaces

Most books of economic analysis dealing with optimization focus on finding necessary conditions for a function defined on a given set which is a subset of a finite dimensional Euclidean space \mathbb{R}^n . These conditions are called first order conditions (in the case of inequality or mixed constraints they are called Kuhn-Tucker conditions). Our main focus of this chapter is the optimization of functions defined on an infinite dimensional vector space (i.e. optimization of functionals).

While we have already encountered linear functionals in Section 6.4, in this section we will consider a broader class of functionals than linear ones; we will consider continuous functionals which are concave (or convex as the

case may warrant). Our main concern is existence theory (i.e. given an optimization problem consisting in maximizing (minimizing) a concave (convex) functional over some feasible set, usually defined by constraints, we want to know whether an optimal solution can be found. Hence we will deal with sufficient conditions. In the second part of the section we also touch upon the problem of finding this optimal solution which means stating the necessary conditions for an optimum.

Example 553 *The types of problems we can consider are: the existence of a Pareto-optimal allocation of an economy with an infinite commodity space; the existence of an optimal solution of an infinite horizon growth model.*

Sufficient Conditions for an Optimal Solution

In this subsection we address the fundamental question “Does a functional have a maximum (or minimum) on a given set?” In Chapter 4, we proved a very important result; the Extreme Value Theorem 262 stated that a continuous function defined on a compact subset of a metric space attains its minimum and maximum. Does this theorem apply to functionals (functions whose domain is a subset of an infinite-dimensional vector space)? Clearly the answer is yes since a vector space is a metric space and dimensionality is not mentioned in the theorem at all. Consider the following example.

Example 554 *Let a functional f be defined on $C([0, 1])$ by $f(x) = \int_0^{\frac{1}{2}} x(t) dt - \int_{\frac{1}{2}}^1 x(t) dt$. We want to solve the optimization problem $\max f(x)$ subject to $\|x\| \leq 1$. To establish continuity of $f(x)$, we need only establish boundedness since $f(x)$ is a linear functional and Theorem 511 establishes that bounded-*

ness is sufficient (and also necessary) for continuity. Hence

$$\begin{aligned}
 |f(x)| &= \left| \int_0^{\frac{1}{2}} x(t) dt - \int_0^{\frac{1}{2}} x(t) dt \right| \\
 &\leq \int_0^{\frac{1}{2}} |x(t)| dt + \int_{\frac{1}{2}}^1 |x(t)| dt \\
 &= \int_0^1 |x(t)| dt \\
 &\leq \int_0^1 \sup_{t \in [0,1]} |x(t)| dt \\
 &= \sup_{t \in [0,1]} |x(t)| \int_0^1 dt \\
 &= \sup_{t \in [0,1]} |x(t)| = \|x\|.
 \end{aligned}$$

Suppose now we want to find the maximum of this continuous functional on the closed unit ball in $C([0, 1])$. But the maximum cannot be attained. Why? Our problem means maximizing the shaded area. See Figure 6.6.2.??? The steeper the middle part of $x(t)$ is the larger is the area. But this line cannot be vertical because $x(t)$ wouldn't be a function. The steepest line clearly doesn't exist.

Exercise 6.6.1 Show the non-existence of a maximum in Example 554 rigorously. Hint: Use the geometric insight provided above.

What went wrong with establishing a maximum in Example 554? We have a continuous functional on a closed unit ball that doesn't attain its maximum. Recall however by Example 459 that a closed unit ball is not a compact set in $C([0, 1])$. In fact there is a theorem saying that a closed unit ball in a normed vector space X is compact if and only if X is finite dimensional.⁸

If a "nice" set like a closed unit ball is not compact, then compactness must be an extremely restrictive assumption in infinite dimensional vector spaces. And it really is. Compact sets in infinite dimensional vector spaces doesn't contain interior points. Thus the Extreme Value Theorem is practically unusable in optimizing functionals.

⁸See Rudin ????

If Example 554 were formulated in $L_\infty(0, 1)$, the maximum would be attained by the discontinuous function

$$x(t) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases} .$$

$L_\infty(0, 1)$ is also infinite dimensional. Is this a contradiction to what we claimed above? No. There are many optimizing problems in infinite dimensional vector spaces that attain optima but an optimum cannot be guaranteed by the assumptions of continuity and compactness. To this end, we will introduce a new type of convergence in a vector space X . In terms of this new convergence we'll define a new type of continuity and compactness so that the collection of these "new types" of compact sets is much broader than the collection of original compact sets. In particular, we will identify a class of such vector spaces in which the closed unit ball is "weakly" compact.

Semicontinuous and concave functionals

Before introducing this "new type" of convergence, we define certain properties of functionals. The concept of convexity and concavity for functionals is analogous to the one for functions.

Definition 555 Let K be a convex subset of a normed vector space X . A functional $f : K \rightarrow \mathbb{R}$ is called: (i) **Concave** if for any $u, v \in K$ and for any $\alpha \in [0, 1]$, $f(\alpha u + (1 - \alpha)v) \geq \alpha f(u) + (1 - \alpha)f(v)$; (ii) **Convex** if $f(\alpha u + (1 - \alpha)v) \leq \alpha f(u) + (1 - \alpha)f(v)$

Exercise 6.6.2 Show that f is concave iff $-f$ is convex.

Exercise 6.6.3 Verify that the functional $f(x) = \int_0^1 (x^2(t) + |x(t)|) dt$ defined on $L_2[0, 1]$ is convex.

Next we introduce the concept of semicontinuity of functionals (or functions as the case may be). Why don't we simply use continuity? Recall we used the assumption of continuity in the Extreme Value theorem to guarantee the existence of *both* a maximum *and* a minimum. Here we will show that the assumption can be weakened at the cost of guaranteeing *either* a maximum *or* a minimum.

Definition 556 A functional f defined on a normed vector space X is said to be: (i) **upper semicontinuous at x_0** if given $\varepsilon > 0$, there is a $\delta > 0$ such that $f(x) - f(x_0) < \varepsilon$ for $\|x - x_0\| < \delta$; (ii) **lower semicontinuous at x_0** if $f(x_0) - f(x) < \varepsilon$ for $\|x - x_0\| < \delta$.

Exercise 6.6.4 Show that f is usc iff $-f$ is lsc.

Exercise 6.6.5 Show that f is continuous at x_0 if f is both usc and lsc at x_0 .

Exercise 6.6.6 A sequence version definition of semicontinuity is the following: (i) f is usc at x_0 if for any sequence $\langle x_n \rangle$ converging to x_0 , $\limsup_{n \rightarrow \infty} f(x_n) \leq f(x_0)$; (ii) f is lsc at x_0 if $\liminf_{n \rightarrow \infty} f(x_n) \geq f(x_0)$. Show that the sequence definition is equivalent to that in 556.

Now the Extreme Value Theorem can be reformulated:

Theorem 557 An upper (lower) semicontinuous functional f on a compact subset K of a normed vector space X achieves a maximum (minimum) on K .

Proof. Let $M = \sup_{x \in K} f(x)$ (M may be ∞). There is a sequence $\langle x_n \rangle$ from K such that $f(x_n) \rightarrow M$. Since K is compact there is a convergent subsequence $\langle x_{n_k} \rangle \rightarrow x_0 \in K$. Clearly, $f(x_{n_k}) \rightarrow M$ and since f is usc, $f(x_0) \geq \limsup_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} f(x_{n_k}) = M$. Because $x_0 \in K$, $f(x_0)$ must be finite and because M is the supremum on K , $f(x_0) = M$. Hence f attains a maximum at $x_0 \in K$. ■

Hereafter, we will formulate our optimization problem in terms of maximization (i.e. given a functional f defined on a subset K of a normed vector space, find $\max_{x \in K} f(x)$). In this case the underlying assumptions for f are upper semicontinuity and concavity. The problem of finding $\min_{x \in K} g(x)$ where g is lower semicontinuous and convex can be transformed to maximizing one by substitution $f = -g$ because if g is lsc and convex, then $-g$ is usc and concave (see Exercise 6.6.2 and 6.6.4).

Weak convergence

We assume that X is a complete normed vector space (i.e. a Banach space).

Definition 558 Let $\langle x_n \rangle$ be a sequence of elements in X . We say that $\langle x_n \rangle$ **converges weakly** to $x_0 \in X$ if for every continuous linear functional $f \in X^*$ we have $\langle f, x_n \rangle \rightarrow \langle f, x_0 \rangle$. We denote weak convergence with “ \rightharpoonup ” (rather than the standard “ \rightarrow ”) and use notation $\langle x_n \rangle \rightharpoonup x_0$.

Since $\langle f, x_n \rangle$ is a value of the functional f at point x_n , then $\langle f, x_n \rangle_{n=1}^{\infty}$ is the sequence of real numbers. It is easy to prove that weak convergence has usual properties namely.

Exercise 6.6.7 Show that: (i) If $\langle x_n \rangle \rightharpoonup x_0$ and $\langle y_n \rangle \rightharpoonup y_0$, then $\langle x_n + y_n \rangle \rightharpoonup x_0 + y_0$; (ii) Let $\langle \lambda_n \rangle$ is a sequence of real numbers. If $\lambda_n \rightarrow \lambda_0$ and $\langle x_n \rangle \rightharpoonup x_0$, then $\lambda_n x_n \rightharpoonup \lambda_0 x_0$; (iii) If $x_n \rightarrow x_0$, then $\langle x_n \rangle \rightharpoonup x_0$; (iv) a weakly convergent sequence has a unique limit. Hint for (iv): Apply the corollary to the Hahn Banach Theorem to $\langle f, x - y \rangle = 0$ for each $f \in X^*$.

Since there are now two types of convergence defined on X , the original one (i.e. with respect to its norm) is sometimes called strong convergence as opposed to (the newly introduced) weak convergence. Property (iii) of the exercise states that if a sequence converges strongly, then it also converges weakly. This statement cannot be reversed in general. That means there are sequences that converge weakly but not strongly. We will see this in the following set of examples where we demonstrate weak convergence in some Banach spaces.

Example 559 In the finite dimensional vector space \mathbb{R}^n , strong and weak convergence coincides (i.e. for $x_n \in \mathbb{R}^n$, $\langle x_n \rangle \rightharpoonup x_0$ iff $\langle x_n \rangle \rightarrow x_0$). To see this, let $e^1 = (1, 0, \dots, 0)$, $e^2 = (0, 1, \dots, 0)$, $e^n = (0, 0, \dots, 1)$, and $\langle x_n \rangle \rightarrow x_0$. Then by definition $\langle f, x_n \rangle \rightarrow \langle f, x_0 \rangle$ where f is a continuous linear functional on \mathbb{R}^n . By Theorem 529 we know that each functional f is represented by a scalar product i.e. given f there exists an element b of \mathbb{R}^n s.t. $\langle f, x \rangle = \langle b, x \rangle$, $\forall x \in X$. (Just reminder that $\langle f, x \rangle$ denotes the value of the functional f at x (i.e. $f(x)$) and $\langle b, x \rangle$ is the scalar product of b and x). If we substitute e^i for b we have $\langle e^i, x_n \rangle = 0x_n^1 + \dots + 1x_n^i + \dots + 0x_n^n = x_n^i \rightarrow x_0^i$, $\forall i = 1, 2, \dots, n$ (i.e. the i th component of the vector x_n tends to the i th component of x_0). Thus weak convergence in \mathbb{R}^n means convergence by components. But Theorem 223 says that then $\langle x_n \rangle \rightarrow x_0$ with respect to the norm that means strongly.

Example 560 Let $X = \ell_2$ and $\langle x_n \rangle \rightharpoonup x_0$. Then as in Example 559 $\langle x_n, e^i \rangle = x_n^i \rightarrow \langle x_0, e^i \rangle = x_0^i$, $\forall i = 1, 2, \dots$. Thus weak convergence in ℓ_2 means that the i -th component of $\langle x_n \rangle$ converges to the i -th component of x_0 . But as Example 234 shows this doesn't imply strong convergence in ℓ_2 .

Example 561 Let $X = \mathcal{C}([a, b])$. It can be shown that weak convergence of a sequence of continuous functions $\langle x_n \rangle \rightharpoonup x_0$ means that: (i) $\langle x_n \rangle$ is uniformly bounded (i.e. $\exists B$ such that $|x_n(t)| \leq B$ for all $n = 1, 2, \dots$ and all $t \in [a, b]$); and (ii) $\langle x_n \rangle \rightarrow x_0$ pointwise on $[a, b]$ (i.e. $\forall t \in [a, b]$, $\langle x_n(t) \rangle \rightarrow x_0(t)$ (as a sequence of real numbers)). Thus weak convergence in $\mathcal{C}([a, b])$ is pointwise convergence (we can say convergence by components) whereas strong convergence (convergence with respect to the sup norm) is uniform. As Examples 166 and 167 show these two don't always coincide.

Using weak convergence allows us to define weak closedness, weak compactness, and weak continuity (or semicontinuity). We do it the same way we did in Chapter 4 where all these notions were defined in terms of sequences.

Definition 562 A subset $K \subset X$ is **weakly closed** if for any sequence $\langle x_n \rangle$ of elements from K that converges weakly to x_0 (i.e. $\langle x_n \rangle \rightharpoonup x_0$), then $x_0 \in K$.

What is the relation between strong and weak closedness? While one would expect that if a set is strongly closed then it is weakly closed, actually the reverse is true.

Theorem 563 If K is weakly closed then K is (strongly) closed.

Proof. Let $\langle x_n \rangle \subset K$ and $\langle x_n \rangle \rightarrow x_0$. then $\langle x_n \rangle \rightharpoonup x_0$ and because K is weakly closed $\implies x_0 \in K$. Hence K is (strongly) closed. ■

To see that Theorem 563 cannot be reversed, we present the following example.⁹

Example 564 Let $M \subset \ell_2$ where $M = \{ \langle e^i \rangle_{i=1}^\infty, e^i = (0, 0, \dots, 1, 0, \dots), i = 1, 2, \dots \}$. M is closed (why?) but it is not weakly closed because $\langle e^i \rangle \rightharpoonup \langle 0 \rangle$ and $\langle 0 \rangle \notin M$.

⁹We cannot give an example in \mathbb{R}^n because in finite dimensional space weak closedness and closedness, of course, coincide.

Theorem 563 and Example 564 say that weak closedness is a stronger assumption than strong closedness. Thus one should be careful in drawing conclusions.

Definition 565 *A set $K \subset X$ is **weakly compact** if every infinite sequence from K contains a weakly convergent subsequence.*

This definition 192 of sequential compactness is equivalent to the standard definition of compactness by Theorem 193 in metric spaces. After our experience with closedness, one may wonder how weak and strong compactness are related. But if weak compactness were stronger assumption than strong compactness (as in the case of closedness) there would be fewer weakly compact sets than compact sets. Then the whole idea of building the weak topology would be useless because the main purpose of the introduction of the weak topology is to make closed unit balls (weakly) compact. Fortunately, it is not the case.

Theorem 566 *If $K \subset X$ is (strongly) compact then it is weakly compact.*

Proof. Let $\langle x_n \rangle$ be a sequence in K . Because K is compact then there is a convergent subsequence $\langle x_{n_k} \rangle$ and $x_0 \in K$ such that $\langle x_{n_k} \rangle \rightarrow x_0$. But strong convergence implies weak convergence so that $\langle x_{n_k} \rangle \rightharpoonup x_0$. ■

Theorem 566 cannot be reversed as the next example shows.

Example 567 *Let $K \subset \ell_2$ where $K = \{ \langle e^i \rangle_{i=0}^\infty, e_0 = (0, 0, \dots, 0, \dots), e^i = (0, \dots, 1, 0, \dots) \}$ (note that $K = M \cup \{ \langle 0 \rangle \}$ when M is from Example 564). K is weakly compact because any sequence from K contains a weakly convergent subsequence (To see this note that since $\langle x_n \rangle \rightharpoonup x_0$, we have $x_n^i \rightarrow x_0^i \forall i = 1, 2, \dots, \langle x_n^i \rangle \subset \{0, 1\}$ and $\{0, 1\}$ is compact in \mathbb{R} . Then there is x_0^i and $\langle x_{n_k}^i \rangle$ such that $\langle x_{n_k}^i \rangle \rightarrow x_0^i$. Then $x_0 = \langle x_0^i, x_0^2, \dots, x_0^i, \dots \rangle$ is the point such that $\langle x_{n_k}^i \rangle \rightharpoonup x_0$.) But K is not compact because the distance between any two elements of $K \setminus \{ \langle 0 \rangle \}$ is $\sqrt{2}$. Hence there doesn't exist a convergent subsequence (with respect to the norm $\|\cdot\|_2$).*

Theorem 568 *If M is weakly compact then M is weakly closed.*

Exercise 6.6.8 *Prove Theorem 568.*

Definition 569 Let $M \subset X$ and f be a functional defined on M . We say that f is **weakly upper semicontinuous (usc)** on M if for any $x_0 \in M$ and any $\langle x_n \rangle_{n=1}^{\infty} \subset M$ such that $x_n \rightharpoonup x_0$, then $f(x_0) \geq \lim_{n \rightarrow \infty} \sup f(x_n)$.

One can define weakly lower semicontinuity and weak continuity of functionals in an analogous manner.

Again, there is a question about whether the assumption of weak upper semicontinuity of a functional is more restrictive than (strong) upper semicontinuity.

Theorem 570 If f is a weakly usc functional on M , then f is usc.

Proof. Let $x_0 \in M$ and $\langle x_n \rangle \subset M$ such that $\langle x_n \rangle \rightarrow x_0$, then $x_n \rightharpoonup x_0$ and because f is weakly usc then $\lim_{n \rightarrow \infty} \sup f(x_n) \leq f(x_0)$ so that f is usc. ■

The converse is not true as the next example shows.

Example 571 Let $X = L_2[0, 1]$ and $f(a) = 1 + \int_0^1 a^2(x) dx$. This functional is continuous (and hence usc) but is not weakly usc.

Exercise 6.6.9 Show the functional in Example 571 is continuous but not weakly usc.

Now we are ready to prove an important theorem which is analogous to the Extreme Value Theorem 262 but uses the concept of the weak topology. A weak topology on X is a topology built in terms of weak convergence instead of (strong) convergence.

Theorem 572 Let K be a non-empty weakly compact subset of a Banach space X . Let f be a weakly upper semicontinuous functional on K . Then f attains its maximum on K . That is, $\exists x_0 \in K$ such that

$$f(x_0) = \sup_{x \in K} f(x)$$

Proof. By the supremum property, $\exists \langle x_n \rangle \subset K$ such that $\lim_{n \rightarrow \infty} f(x_n) = \sup_{x \in K} f(x)$. Since K is weakly compact, there exists a subsequence $\langle x_{n_k} \rangle$ and $x_0 \in K$ such that $x_{n_k} \rightharpoonup x_0$. Because f is weakly usc then

$$f(x_0) \geq \lim_{n \rightarrow \infty} \sup f(x_{n_k}) = \lim_{n \rightarrow \infty} f(x_n) = \sup_{x \in K} f(x) .$$

Obviously $f(x_0) \leq \sup_{x \in K} f(x)$ because $x_0 \in K$. Thus combining these two inequalities we have

$$f(x_0) = \sup_{x \in K} f(x).$$

■

Comparing the assumptions of Theorem 572 with the Extreme Value Theorem 262 we see that in Theorem 572 one assumption is weaker (that being compactness) and one is stronger (that being semicontinuity). The problem with this theorem is that it has basically non-verifiable assumptions. How should one check weak compactness and weak semicontinuity in an infinite dimensional space? Our next step is to find sufficient and at the same time verifiable assumptions that would guarantee weak compactness of a set K and weak use of a functional f .

Let's start with weak compactness. We already know that (strong) compactness is sufficient for weak compactness but we also know that it is too restrictive in infinite dimensional vector spaces. In order for a set K to be weakly compact it has to be weakly closed (see Theorem 568). First we examine the conditions for a set K to be weakly closed. Theorem 563 says that it must be closed but that's not sufficient (see Example 564). There are however quite simple assumptions that guarantee weak closedness of a set K .

Theorem 573 *If $K \subset X$ is closed and convex then it is weakly closed.*

Proof. Let $\langle x_n \rangle \subset K$ and $\langle x_n \rangle \rightharpoonup x_0$. Then we need to show that $x_0 \in K$. Assume the contrary, that is $x_0 \notin K$. Then by Corollary 550 of the Separation Theorem, there exists a non-zero continuous linear functional f such that $\langle f, x_0 \rangle < \inf_{x \in K} \langle f, x \rangle$. Let $\langle f, x_0 \rangle = c$ and $\inf_{x \in K} \langle f, x \rangle = d$ in which case $c < d$. Because f is a linear continuous functional we have $d \leq \lim_{n \rightarrow \infty} \langle f, x_n \rangle = \langle f, x_0 \rangle = c < d$. Hence $d < d$ which is the desired contradiction. ■

Theorem 573 says that closedness and convexity are sufficient assumptions for weak closedness. However, we are looking for sufficient assumptions for weak compactness. To make further progress, we have to restrict attention to certain classes of normed vector spaces. In Section 6.4.2 we defined a reflexive space as a space for which $X^{**} = X$ (see Definition 533). We showed that, for example, \mathbb{R}^n , ℓ_p , L_p for $1 < p < \infty$ are reflexive whereas ℓ_1 , ℓ_∞ , L_1 , L_∞ , $\mathcal{C}([a, b])$ are not reflexive. From here on we will consider only reflexive, normed vector spaces. Our next result is basically a Heine-Borel theorem for infinite dimensional spaces.

Theorem 574 (Eberlein-Šmuljan) *A Banach space X is reflexive iff any bounded weakly closed set $K \subset X$ is weakly compact.*

Proof. Can be found in Aliprantis (1985, Theorem 10.13, p.156, Positive Operators). ■

Thus in a reflexive Banach space, weak closedness and boundedness are sufficient assumptions for weak compactness and in any Banach space closedness and convexity are sufficient assumptions for weak closedness. Putting all these together we have sufficient assumptions for a set K to be weakly compact in a reflexive Banach space X .

Theorem 575 *Let X be a reflexive Banach space and $K \subset X$. If K is closed, bounded and convex, then K is weakly compact.*

Proof. Combine Theorems 573 and 574. Notice that all the assumptions of the theorem are verifiable. ■

Corollary 576 *In a reflexive space X , the closed unit ball is a weakly compact set.*

Proof. $B_1(0) = \{x \in X : \|\cdot\| \leq 1\}$ is a closed, bounded, and convex subset of a reflexive space X . Hence by Theorem 575. ■

Let's turn now to the assumption of a "weakly upper semicontinuous functional" and try to break it into verifiable parts. We first prove a lemma that gives us a necessary and sufficient condition for a functional f to be weakly upper semicontinuous.

Lemma 577 *Let X be a Banach space and $K \subset X$ be weakly closed. Let a functional f be defined on M . Then f is weakly upper semicontinuous on K iff $\forall a \in \mathbb{R}, E(a) = \{v \in K : f(v) \geq a\}$ is weakly closed.*

Proof. (\implies) Let f be weakly usc on M , $a \in \mathbb{R}$, and $\langle x_n \rangle_{n=1}^\infty \subset E(a)$ such that $x_n \rightharpoonup x_0 \in M$. Then $f(x_0) \geq \limsup_{n \rightarrow \infty} f(x_n) \geq a$ (because $x_n \in E(a) \forall n$). Hence $x_0 \in E(a)$ and thus $E(a)$ is weakly closed.

(\impliedby) By contradiction. Let $a \in \mathbb{R}$, $E(a)$ be weakly closed, but f not weakly usc. Then there exists $x_0 \in M$ and $\langle x_n \rangle_{n=1}^\infty \subset M$ such that $\langle x_n \rangle \rightharpoonup x_0$ and $\limsup_{n \rightarrow \infty} f(x_n) > f(x_0)$. Choose $a \in \mathbb{R}$ such that $\limsup_{n \rightarrow \infty} f(x_n) > a > f(x_0)$. Then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \in E(a)$, $k = 1, 2, \dots$. Because $E(a)$ is weakly closed and

$\langle x_{n_k} \rangle \rightharpoonup x_0$ then $x_0 \in E(a)$. Thus $f(x_0) \geq a > f(x_0)$, which is a contradiction. ■

Thus f is weakly usc if $E(a)$ is weakly closed. But by Theorem 573 we know that if a set is closed and convex, then it is weakly closed. When is $E(a)$ closed? Since $E(a)$ is just the inverse image of the interval $[a, \infty)$ (i.e. $E(a) = \{v \in K : f(v) \geq a\} = f^{-1}([a, \infty))$), if f is continuous then the inverse image of a closed set is closed (by a modification of Theorem [200]). Thus if f is continuous, then $E(a)$ is closed. When is $E(a)$ convex?

Exercise 6.6.10 *Show that if f is concave, then $E(a)$ is convex.*

Combining these two results we have sufficient conditions for weak upper semicontinuity.

Theorem 578 *A continuous, concave functional f defined on a closed, convex set $K \subset X$ is weakly upper semicontinuous.*

Now when we combine Theorems 575 and 578 with Theorem 572 we get a theorem that guarantees the existence of a maximum and all its assumptions are "easily" verifiable.

Theorem 579 *Let K be a non-empty, convex, closed and bounded subset of a reflexive Banach space. Let f be a continuous and concave functional defined on \mathbb{R} . Then f attains its maximum on K (i.e. $\exists x^* \in K$ such that $f(x^*) = \sup_{x \in K} f(x)$).*

First we note that minimization requires convexity of the functional f instead of concavity while all other assumptions are the same. Second, we want to stress that this is a nonlinear optimization problem. This means the functional f doesn't have to be linear (which is quite restrictive). The functional f simply has to be continuous and concave in the case of maximization and continuous and convex in the case of minimization.

Let us summarize what we have done in this section. In infinite dimensional vector spaces the original Extreme Value Theorem 262 (requiring semicontinuity of a functional and compactness of a set) which guarantees the existence of an optimum cannot be used since the assumption of compactness is too stringent (compact sets don't contain interior points). By introducing the weak topology on X we define weak semicontinuity (an assumption that is stronger than the continuity) and weak compactness (an assumption the

is weaker than compactness). We also must enlist the extra assumptions of concavity (or convexity) of a functional and reflexivity of the space X . Then we showed that with these modified assumptions an analogue of the Extreme Value Theorem holds and this version "covers" more optimization problems (e.g. optimizing over unit balls).

6.6.2 Dynamic Programming

An important and frequently used example of operators is dynamic programming. In infinite horizon problems, dynamic programming turns the problem of finding an infinite sequence (or plan) describing the evolution of a vector of (endogenous) state variables into simply choosing a single vector value for the state variables and finding the solution to a functional equation.

More specifically, suppose the primitives of the problem are as follows. Let X denote the set of possible values of (endogenous) state variables with typical element x . We will assume that $X \subset \mathbb{R}^n$ is compact and convex. Let $\Gamma : X \rightrightarrows X$ be the constraint correspondence describing feasible values for the endogenous state variable. We will assume $\Gamma(x)$ is nonempty, compact-valued, and continuous. Let $G = \{(x, y) \in X \times X : y \in \Gamma(x)\}$ denote the graph of Γ . Let $r : G \rightarrow \mathbb{R}$ denote the per-period objective or return function which we assume is continuous. Finally let $\beta \in (0, 1)$ denote the discount factor. Thus, the "givens" for the problem are X, Γ, r, β . In this section we establish under what conditions solutions to the functional equation (FE)

$$v(x_0) = \max_{y \in \Gamma(x_0)} r(x_0, y) + \beta v(y) \quad (\text{FE})$$

"solve" the sequence problem that is our ultimate objective

$$\max_{\langle x_{t+1} \rangle_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, x_{t+1}) \quad (\text{SP})$$

$$s.t. \ x_{t+1} \in \Gamma(x_t), \forall t, \text{ and } x_0 \text{ given.}$$

First we should establish what we mean by "solve". To begin with, we need to know that (SP) is well defined. That is, we must establish conditions under which the feasible set is nonempty and the objective function is well defined for all points in the feasible set. To accomplish this, we need to

introduce some more notation. Call the sequence $\langle x_t \rangle$ a **plan**. Given $x_0 \in X$, let

$$F(x_0) = \{ \langle x_t \rangle_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t), t = 0, 1, \dots \}$$

be the set of **feasible plans** from x_0 with typical element $\chi = (x_0, x_1, \dots) \in F(x_0)$. Let $\varphi_k : F(x_0) \rightarrow \mathbb{R}$ be given by

$$\varphi_k(\chi) = \sum_{t=0}^k \beta^t r(x_t, x_{t+1})$$

which is simply the discounted partial sum of returns from any feasible plan χ . Finally, let $\varphi : F(x_0) \rightarrow \mathbb{R}$ be given by $\varphi(\chi) = \lim_{k \rightarrow \infty} \varphi_k(\chi)$. While φ_k is obviously well defined, φ may not be since there may be χ such that $\varphi(\chi) = \pm\infty$.¹⁰ The assumption that $\Gamma(x) \neq \emptyset, \forall x \in X$ ensures that $F(x_0)$ is nonempty for all $x_0 \in X$. The assumptions that X is compact and Γ is compact-valued and continuous guarantees $|r(x_t, x_{t+1})| \leq M < \infty$ so that since $\beta \in (0, 1)$ we have $|\varphi(\chi)| \leq \frac{M}{(1-\beta)} < \infty, \forall \chi \in F(x_0), \forall x_0$. Hence SP is well defined and we can define the function $v^* : X \rightarrow \mathbb{R}$ given by

$$v^*(x_0) = \max_{\chi \in F(x_0)} \varphi(\chi) \quad (\text{SP}') \tag{6.15}$$

which is just (SP). Thus by “solve” we mean that $v^*(x_0)$ defined in (SP') is equal to $v(x_0)$ defined in (FE).

Before providing conditions under which a solution to (FE) implies a “solution” to (SP), we note the following consequences of the maximum function defined in (SP'). In particular, by Definition 96 we have

$$v^*(x_0) \geq \varphi(\chi), \forall \chi \in F(x_0) \tag{6.16}$$

and $\forall \varepsilon > 0$,

$$v^*(x_0) < \varphi(\chi) + \varepsilon, \text{ for some } \chi \in F(x_0). \tag{6.17}$$

Similarly, v satisfies (FE) if

$$v(x) \geq r(x, y) + \beta v(y), \forall y \in \Gamma(x) \tag{6.18}$$

and $\forall \varepsilon > 0$,

$$v(x) < r(x, y) + \beta v(y) + \varepsilon, \text{ for some } y \in \Gamma(x). \tag{6.19}$$

Now we are ready to prove our main result that if we have a solution to (FE), then we have a solution to (SP).

¹⁰More generally Stokey and Lucas (1989) consider φ in the extended reals.

Theorem 580 *If v is a solution to (FE) and satisfies*

$$\lim_{k \rightarrow \infty} \beta^k v(x_k) = 0, \forall \langle x_t \rangle \in F(x_0), \forall x_0 \in X, \quad (6.20)$$

then $v = v^$.*

Proof. It suffices to show that if (6.18) and (6.19) hold, then (6.16) and (6.17) are satisfied. Inequality (6.18) implies that $\forall \chi \in F(x_0)$,

$$\begin{aligned} v(x_0) &\geq r(x_0, x_1) + \beta v(x_1) \\ &\geq r(x_0, x_1) + \beta [r(x_1, x_2) + \beta v(x_2)] \\ &\geq \varphi_k(\chi) + \beta^{k+1} v(x_{k+1}), k = 1, 2, \dots \end{aligned}$$

Taking the limit as $k \rightarrow \infty$ and using (6.20), we have (6.16).

Fix $\varepsilon > 0$ and choose $\langle \delta_t \rangle_{t=1}^\infty \subset \mathbb{R}_+$ such that $\sum_{t=1}^\infty \beta^{t-1} \delta_t \leq \varepsilon$. Inequality (6.19) implies there exists $x_1 \in \Gamma(x_0)$, $x_2 \in \Gamma(x_1)$, ... so that

$$v(x_t) \leq r(x_t, x_{t+1}) + \beta v(x_{t+1}) + \delta_{t+1}, t = 0, 1, \dots$$

Then

$$\begin{aligned} v(x_0) &\leq r(x_0, x_1) + \beta v(x_1) + \delta_1 \\ &\leq r(x_0, x_1) + \beta [r(x_1, x_2) + \beta v(x_2) + \delta_2] + \delta_1 \\ &\leq \varphi_k(\chi) + \beta^{k+1} v(x_{k+1}) + \sum_{t=1}^k \beta^{t-1} \delta_t, k = 1, 2, \dots \end{aligned}$$

Taking the limit as $k \rightarrow \infty$ and using (6.20), we have (6.17). ■

Next we establish that a feasible plan which satisfies (FE) is an optimal plan in the sense of (SP).

Theorem 581 *Let $\chi^* \in F(x_0)$ be a feasible plan from x_0 which satisfies the functional equation*

$$v^*(x_t^*) = r(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*) \quad (6.21)$$

with

$$\lim_{t \rightarrow \infty} \max \beta^t v^*(x_t^*) \leq 0, \quad (6.22)$$

then χ^ attains the maximum in (SP) for initial state x_0 .*

Proof. It follows by an induction on (6.21) that

$$\begin{aligned} v^*(x_0) &= r(x_0, x_1^*) + \beta v^*(x_1^*) \\ &= r(x_0, x_1^*) + \beta [r(x_1^*, x_2^*) + \beta v^*(x_2^*)] \\ &= \varphi_k(\chi^*) + \beta^{k+1} v^*(x_{k+1}^*), k = 1, 2, \dots \end{aligned}$$

Then as $k \rightarrow \infty$ using (6.22) we have $v^*(x_0) \leq \varphi(\chi^*)$. Since $\chi^* \in F(x_0)$, we have $v^*(x_0) \geq \varphi(\chi^*)$ by (6.16). Thus χ^* attains the maximum. ■

Now that we have established that a solution v to (FE) is a solution to the (SP) problem we are interested in, we set out to establish the existence of a solution to (FE). Since r is a real valued, bounded, and continuous function, it makes sense to look for solutions in the space of continuous bounded functions $\mathcal{C}(X)$ with the sup norm $\|v\| = \sup \{|v(x)|, x \in X\}$ studied in section 6.1. Furthermore, given a solution $v \in \mathcal{C}(X)$, we can define the policy correspondence $\gamma : X \rightarrow X$ by

$$\gamma(x) = \{y \in \Gamma(x) : v(x) = r(x, y) + \beta v(y)\}. \quad (6.23)$$

This generates a plan since given x_0 , we have $x_1 = \gamma(x_0)$, $x_2 = \gamma(x_1)$, ...

To this end, we define an operator $T : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ given by

$$(Tf)(x) = \max_{y \in \Gamma(x)} [r(x, y) + \beta f(y)]. \quad (6.24)$$

In this case (FE) becomes $v = Tv$. That is, all we must establish is that T has a unique fixed point in $\mathcal{C}(X)$.

Before actually doing that, we provide a simple set of sufficient conditions to establish a given operator is a contraction.

Lemma 582 (Blackwell's sufficient conditions for a contraction) *Let $X \subset \mathbb{R}^n$ and $\mathfrak{B}(X, \mathbb{R})$ be the space of bounded functions $f : X \rightarrow \mathbb{R}$ with the sup norm. Let $T : \mathfrak{B}(X, \mathbb{R}) \rightarrow \mathfrak{B}(X, \mathbb{R})$ be an operator satisfying: (i) (monotonicity) $f, \tilde{f} \in \mathfrak{B}(X, \mathbb{R})$ and $f(x) \leq \tilde{f}(x)$ implies $(Tf)(x) \leq (T\tilde{f})(x), \forall x \in X$; (ii) (discounting) $\exists \rho \in (0, 1)$ such that $[T(f + a)](x) \leq (Tf)(x) + \rho a, a \geq 0, x \in X$.¹¹ Then T is a contraction with modulus ρ .*

Proof. For any $f, \tilde{f} \in \mathfrak{B}(X, \mathbb{R})$, $f \leq \tilde{f} + \|f - \tilde{f}\|$ where we write $f \leq \tilde{f}$ if $f(x) \leq \tilde{f}(x), \forall x \in X$. Then

$$Tf \leq T\left(\tilde{f} + \|f - \tilde{f}\|\right) \leq T\tilde{f} + \rho \|f - \tilde{f}\|$$

¹¹Note $(f + a)(x) = f(x) + a$.

where the first inequality follows from (i) and the second from (ii). Reversing the inequality gives $T\tilde{f} \leq Tf + \rho \|\tilde{f} - f\|$. Combining both inequalities gives $\|Tf - T\tilde{f}\| \leq \rho \|f - \tilde{f}\|$. ■

Now we have the second main theorem of this section.

Theorem 583 *In $(\mathcal{C}(X), \|\cdot\|)$, T given in (6.24) has the following properties: $T(\mathcal{C}(X)) \subset \mathcal{C}(X)$; $Tv = v \in \mathcal{C}(X)$; and $\forall v_0 \in \mathcal{C}(X)$,*

$$\|v - T^n v_0\| \leq \frac{\beta^n}{(1 - \beta)} \|Tv_0 - v_0\|, n = 0, 1, 2, \dots \quad (6.25)$$

Furthermore, given v , the optimal policy correspondence $\gamma : X \rightarrow X$ defined in (6.23) is compact-valued and u.h.c.

Proof. For each $f \in \mathcal{C}(X)$ and $x \in X$, the problem in (6.24) is to maximize a continuous function $[r(x, \cdot) + \beta f(\cdot)]$ on a compact set $\Gamma(x)$. Hence, by the Extreme Value Theorem 262, the maximum is attained. Since both r and f are bounded, Tf is also bounded. Since r and f are continuous and Γ , it follows from the Theorem of the Maximum 295 that Tf is continuous. Hence $T(\mathcal{C}(X)) \subset \mathcal{C}(X)$.

It is clear that T satisfies Blackwell's sufficient conditions for a contraction (Lemma 582) since: (i) for $f, \tilde{f} \in \mathcal{C}(X)$ with $f(x) \leq \tilde{f}(x), \forall x \in X$, by T given in (6.24) we have $(Tf)(x) \leq (T\tilde{f})(x) \Leftrightarrow \max_{y \in \Gamma(x)} [r(x, y) + \beta f(y)] \leq \max_{y \in \Gamma(x)} [r(x, y) + \beta \tilde{f}(y)]$; and (ii)

$$\begin{aligned} T(f + a)(x) &= \max_{y \in \Gamma(x)} [r(x, y) + \beta (f(y) + a)] \\ &= \max_{y \in \Gamma(x)} [r(x, y) + \beta f(y)] + \beta a \\ &= (Tf)(x) + \beta a. \end{aligned}$$

Since $\mathcal{C}(X)$ is a complete normed vector space by Theorem 452 and T is a contraction, then T has a unique fixed point $v \in \mathcal{C}(X)$ by the Contraction Mapping Theorem 306 which satisfies (6.25). The properties of γ follow from the Theorem of the Maximum 295. ■

If we want to say more about v (and γ), we need to impose more structure on the primitives. The next theorem illustrates this.

Theorem 584 For each y , let $r(\cdot, y)$ be strictly increasing in each of its first n arguments and let Γ be monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subset \Gamma(x')$. Then v given by the solution to (FE) is strictly increasing.

Proof. Let $\widehat{\mathcal{C}}(X) \subset \mathcal{C}(X)$ be the set of bounded, continuous, nondecreasing functions and $\widetilde{\mathcal{C}}(X) \subset \widehat{\mathcal{C}}(X)$ be the set of bounded, continuous, strictly increasing functions. Since $\widehat{\mathcal{C}}(X)$ is a closed subset of the Banach space $\mathcal{C}(X)$, Corollary 307 and Theorem 583 imply it is sufficient to show $T(\widehat{\mathcal{C}}(X)) \subset \widetilde{\mathcal{C}}(X)$, which is guaranteed by the assumptions on r and Γ . ■

Existence of solutions with unbounded returns

As we mentioned in the introduction, one application of dynamic optimization in infinite dimensional spaces is the growth model. In that case the endogenous state variable is capital, denoted k_t at any point in time $t = 0, 1, 2, \dots$, with $k_t \in \mathbb{R}_+$ and $k_0 > 0$ given. It is typically not the case that we assume k_t lies in a compact set, which is very different from the assumptions of the previous section. That is, the previous section relied heavily on the fact that the return function was bounded (so that we could work in the space of bounded functions).

To address this problem, here we will consider a specific example. There is a linear production technology where output $y_t = \widetilde{A}k_t$, $\widetilde{A} > 0$. Capital depreciates over a period at rate $\delta > 0$. Assume that $\widetilde{A} = A + (1 - \delta) > 0$. A household is risk neutral (i.e. $u(c_t) = c_t$ where c_t denotes consumption at time t) and discounts the future at rate β . Assume that $\beta^{-1} > \widetilde{A}$. Since utility is strictly increasing in consumption, there is no free disposal and the budget constraint implies that $c_t = Ak_t + (1 - \delta)k_t - k_{t+1}$. Hence the household's reward function is given by $r(k_t, k_{t+1}) \equiv u(c_t) = \widetilde{A}k_t - k_{t+1}$.

The problem of the household is to choose a sequence of capital stocks to maximize the present discounted value of future rewards which is just

$$v^*(k_0) = \sup_{\langle k_{t+1} \rangle} \sum_{t=0}^{\infty} \beta^t [\widetilde{A}k_t - k_{t+1}] \quad (\text{SP})$$

$$\text{s.t. } 0 \leq k_{t+1} \leq \widetilde{A}k_t \text{ and } k_0 > 0 \text{ given.}$$

We will attack this problem in several steps.

1. Show that the constraint correspondence $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by $\Gamma(k_t) = \{k_{t+1} \in \mathbb{R}_+ : k_{t+1} \in [0, \tilde{A}k_t]\}$ is nonempty, compact-valued, continuous, and for any $k_t \in \mathbb{R}_+$, $k_{t+1} \in \Gamma(k_t)$ implies $\lambda k_{t+1} \in \Gamma(\lambda k_t)$ for $\lambda \geq 0$. Furthermore, show that

$$\text{for some } \alpha \in (0, \beta^{-1}), k_{t+1} \leq \alpha k_t, \forall k_t \in \mathbb{R}_+ \text{ and } \forall k_{t+1} \in \Gamma(k_t). \quad (6.26)$$

Nonempty: $k_{t+1} = 0 \in \Gamma(k_t)$. **Compact:** By Heine-Borel, it is sufficient to check that Γ is closed and bounded. Boundedness follows since given $k_t, k_{t+1} \in [0, \tilde{A}k_t]$. To see Γ is closed, suppose $\langle x_n \rangle$ is a sequence such that $x_n \in \Gamma(k_t)$ and $x_n \rightarrow x$. But $x_n \in \Gamma(k_t)$ implies $x_n \in [0, \tilde{A}k_t]$ and $x_n \rightarrow x$ implies $x \in [0, \tilde{A}k_t]$. Thus $x \in \Gamma(k_t)$ so that Γ is closed. **Continuous:** One way to establish this is to show Γ is uhc and lhc. On the other hand, it is clear that since the upper endpoint is linear in k_t , it is continuous in k_t and hence $\Gamma(k_t)$ is continuous. **Homogeneity:** If $k_{t+1} \in \Gamma(k_t)$, then $0 \leq k_{t+1} \leq \tilde{A}k_t$. Multiplying by λ implies $0 \leq \lambda k_{t+1} \leq \lambda \tilde{A}k_t = \tilde{A} \lambda k_t$ or $\lambda k_{t+1} \in \Gamma(\lambda k_t)$. **Existence of α :** Since $k_{t+1} \in \Gamma(k_t)$, then $k_{t+1} \leq \tilde{A}k_t$. Hence just take $\alpha = \tilde{A}$. By assumption, $\tilde{A} < \beta^{-1}$ so $\alpha \in (0, \beta^{-1})$.

2. Show that the conditions you proved in part 1 implies that for any k_0 ,

$$k_t \leq \alpha^t k_0,$$

$\forall t$ and for all feasible plans $\langle k_{t+1} \rangle \in F(k_0) = \{\langle k_{t+1} \rangle : k_{t+1} \in \Gamma(k_t), t = 0, 1, \dots\}$, the set of plans feasible from k_0 . To see this, from 1, $k_{t+1} \leq \tilde{A}k_t = \alpha k_t \leq \alpha(\alpha k_{t-1}) = \alpha^2 k_{t-1} \leq \dots \leq \alpha^{t+1} k_0$. Notice that k_t can be growing over time, though at rate less than β^{-1} .

3. Let $G = \{(k_t, k_{t+1}) \in \mathbb{R}_+ \times \mathbb{R}_+ : k_{t+1} \in \Gamma(k_t)\}$. Show that $r : G \rightarrow \mathbb{R}_+$ is continuous and homogeneous of degree one. Show that $\tilde{A}k_t - k_{t+1} \geq 0, \forall t$ and $\exists B \in (0, \infty)$ such that

$$\tilde{A}k_t - k_{t+1} \leq B(k_t + k_{t+1}), \quad \forall (k_t, k_{t+1}) \in G. \quad (6.27)$$

Given that $(k_t, k_{t+1}) \in \mathbb{R}_+ \times \mathbb{R}_+$, (6.27) assures a uniform bound on the ratio of the return function u and the norm of its arguments. **Continuous:** We must show that $\forall \varepsilon > 0, \exists \delta(k_t, k_{t+1}, \varepsilon) > 0$ such that if

$$\sqrt{(k_t^0 - k_t)^2 + (k_{t+1}^0 - k_{t+1})^2} < \delta, \quad (6.28)$$

then

$$\left| \left(\tilde{A}k_t^0 - k_{t+1}^0 \right) - \left(\tilde{A}k_t - k_{t+1} \right) \right| < \varepsilon. \quad (6.29)$$

But

$$\left| \left(\tilde{A}k_t^0 - k_{t+1}^0 \right) - \left(\tilde{A}k_t - k_{t+1} \right) \right| \leq \tilde{A} |k_t^0 - k_t| + |k_{t+1}^0 - k_{t+1}|$$

by the triangle inequality. If (6.28) is satisfied, then

$$\tilde{A} |k_t^0 - k_t| \leq \tilde{A} \sqrt{(k_t^0 - k_t)^2 + (k_{t+1}^0 - k_{t+1})^2} < \tilde{A}\delta$$

$$\text{and } |k_{t+1}^0 - k_{t+1}| \leq \sqrt{(k_t^0 - k_t)^2 + (k_{t+1}^0 - k_{t+1})^2} < \delta.$$

Hence let $\delta = \max\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2\tilde{A}}\right\}$. **Homogeneity:** $r(\lambda k_t, \lambda k_{t+1}) = \tilde{A}\lambda k_t - \lambda k_{t+1} = \lambda [\tilde{A}k_t - k_{t+1}] = \lambda r(k_t, k_{t+1})$. **Nonnegative returns:** Since $k_{t+1} \in [0, \tilde{A}k_t]$, we know $\tilde{A}k_t - k_{t+1} \geq 0, \forall t$. **Boundedness:** Inequality (6.27) is established since

$$\tilde{A}k_t - k_{t+1} \leq \tilde{A}k_t + k_{t+1} \leq \max\{\tilde{A}, 1\} (k_t + k_{t+1}), \forall (k_t, k_{t+1}) \in G$$

where $B = \max\{\tilde{A}, 1\}$.

4. Show that the conditions you proved in the previous parts imply that for any k_0 and $\forall \langle k_{t+1} \rangle \in F(k_0)$,

$$\lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t [\tilde{A}k_t - k_{t+1}]$$

exists. This, along with the prior conditions you have proven, establishes that a solution to (SP) satisfies the functional equation

$$v(k_t) = \sup_{k_{t+1} \in \Gamma(k_t)} [\tilde{A}k_t - k_{t+1}] + \beta v(k_{t+1}). \quad (\text{FE})$$

We start by noting

$$\begin{aligned} \sum_{t=0}^n \beta^t [\tilde{A}k_t - k_{t+1}] &\leq \sum_{t=0}^n \beta^t B [k_t + k_{t+1}] \\ &\leq B \sum_{t=0}^n \beta^t [\alpha^t k_0 + \alpha^{t+1} k_0] \\ &= B k_0 (1 + \alpha) \sum_{t=0}^n (\alpha\beta)^t \\ &\leq B k_0 (1 + \alpha) / (1 - \alpha\beta) \end{aligned}$$

where the first inequality follows from part 3, the second from part 2, and the third since $\alpha\beta < 1$ from part 1. Since $\sum_{t=0}^n \beta^t [\tilde{A}k_t - k_{t+1}]$ is increasing and bounded, the limit exists.

5. Show that v^* defined in (SP) is homogeneous of degree one (i.e. $v^*(\theta k_0) = \theta v^*(k_0)$) and that for some $\eta \in (0, \infty)$, $|v^*(k_0)| \leq \eta k_0$, for any k_0 . To see this, as in part 2, consider $\langle k_{t+1} \rangle \in F(k_0)$ and Let $u(\langle k_{t+1} \rangle) = \lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t [\tilde{A}k_t - k_{t+1}]$. Then $v^*(k_0) = \sup_{\langle k_{t+1} \rangle \in F(k_0)} u(\langle k_{t+1} \rangle)$. For $\theta > 0$, $\theta k_0 \in \mathbb{R}_+$ since \mathbb{R}_+ is a convex cone. Furthermore, $k_1 \in \Gamma(k_0) \Rightarrow \theta k_1 \in \Gamma(\theta k_0)$ as established in part 1. Continuing in this fashion we can show that $\forall \langle k_{t+1} \rangle \in F(k_0)$, $\langle \theta k_{t+1} \rangle \in F(\theta k_0)$.

Homogeneity: For $\theta > 0$,

$$\begin{aligned} v^*(\theta k_0) &= \sup_{\langle \theta k_{t+1} \rangle \in F(\theta k_0)} u(\langle \theta k_{t+1} \rangle) \\ &= \sup_{\langle \theta k_{t+1} \rangle \in F(\theta k_0)} \left\{ \lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t [\tilde{A}\theta k_t - \theta k_{t+1}] \right\} \\ &= \sup_{\langle \theta k_{t+1} \rangle \in F(\theta k_0)} \left\{ \lim_{n \rightarrow \infty} \theta \sum_{t=0}^n \beta^t [\tilde{A}k_t - k_{t+1}] \right\} \\ &= \theta v^*(k_0). \end{aligned}$$

Boundedness: Let $D = B(1 + \alpha)/(1 - \alpha\beta) > 0$. Then $\forall \langle k_{t+1} \rangle \in F(k_0)$, it was shown in part 4 that

$$|u(\langle k_{t+1} \rangle)| = \left| \lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t [\tilde{A}k_t - k_{t+1}] \right| \leq Dk_0.$$

Thus, $\forall \langle k_{t+1} \rangle \in F(k_0)$,

$$|v^*(k_0)| = \left| \sup_{\langle k_{t+1} \rangle \in F(k_0)} u(\langle k_{t+1} \rangle) \right| \leq \sup_{\langle k_{t+1} \rangle \in F(k_0)} |u(\langle k_{t+1} \rangle)| \leq Dk_0.$$

6. As a converse to the above results, next consider seeking solutions to (FE) in the space of functions (denoted $H(\mathbb{R}_+)$) that are continuous, homogeneous of degree 1, and bounded in the sense that if $f \in H(\mathbb{R}_+)$, then $\frac{|f(k_t)|}{k_t} < \infty$. This notion of boundedness is consistent with Definition 522. Endow the space with the operator norm, which by Theorem

518 can be of the form

$$\|f\| = \sup \left\{ \frac{|f(k_t)|}{k_t}, k_t \in \mathbb{R}_+, k_t \neq 0 \right\}. \quad (6.30)$$

We must verify that $\|\cdot\|$ on the vector space $H(\mathbb{R}_+)$ satisfies the properties of a norm and that $(H(\mathbb{R}_+), \|\cdot\|)$ is complete. **Normed Vector Space:** We must show: (a) $\|f\| \geq 0$ with equality iff $f = 0$; (b) $\|af\| = |a| \cdot \|f\|$; and (c) $\|f + f'\| \leq \|f\| + \|f'\|$. **Complete:** Consider any Cauchy sequence $\langle f_n \rangle$ in $H(\mathbb{R}_+)$. For all $x \in \mathbb{R}_+$, $\langle f_n(x) \rangle$ is Cauchy in \mathbb{R} and thus has a limit. Define $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $\forall x \in \mathbb{R}_+$. We must show that $f \in H(\mathbb{R}_+)$ by verifying that (i) f is homogeneous of degree 1; (ii) bounded in the sense that $\frac{|f(x)|}{x} < \infty$; and (iii) f is continuous. Starting with (i), for $\lambda > 0$ and $x \in \mathbb{R}_+$, since $f_n \in H(\mathbb{R}_+) \forall n$ we have $f(\lambda x) = \lim_{n \rightarrow \infty} f_n(\lambda x) = \lim_{n \rightarrow \infty} \lambda f_n(x) = \lambda f(x)$. Next, for (ii) we know that f is bounded since $\langle \|f_n\| \rangle$ is a convergent sequence in \mathbb{R} and thus bounded (i.e. $\|f\| \leq \|f_N\| + 1$ for some $N \in \mathbb{N}$). Finally, for (iii) note that $\langle f_n \rangle \rightarrow f$ uniformly by the Cauchy criterion and that $H(\mathbb{R}_+) \subset C(\mathbb{R}_+)$. But uniform convergence of continuous functions implies the limit function is continuous.

7. Show that for any $v \in H(\mathbb{R}_+)$, $\lim_{n \rightarrow \infty} \beta^t v(k_t) = 0$ which establishes the conditions necessary to prove that a solution to (FE) implies a solution to (SP). It follows directly from parts 2 and 6 that for any $f \in H(\mathbb{R}_+)$,

$$|f(k_t)| \leq k_t \cdot \|f\| \leq \alpha^t k_0$$

so that since $\alpha\beta < 1$, $\lim_{n \rightarrow \infty} \beta^t v(k_t) = 0$.

8. Define an operator T on $H(\mathbb{R}_+)$ by

$$(Tf)(k) = \sup_{k' \in \Gamma(k)} \left[\tilde{A}k - k' + \beta f(k') \right]. \quad (6.31)$$

Show that T maps functions in $H(\mathbb{R}_+)$ to functions in $H(\mathbb{R}_+)$. **Continuity:** Since r and f are continuous by part 3 and $f \in H(\mathbb{R}_+)$, and Γ is compact valued, we know Tf is continuous by the Theorem of the Maximum. **Boundedness:** Since r and f are bounded by part 3 and

$f \in H(\mathbb{R}_+)$, Tf is bounded. **Homogeneity:** For $\theta > 0$,

$$\begin{aligned} (Tf)(\theta k) &= \sup_{\theta k' \in \Gamma(\theta k)} \left[\tilde{A}\theta k - \theta k' + \beta f(\theta k') \right] \\ &= \theta \sup_{\theta k' \in \Gamma(\theta k)} \left[\tilde{A}k - k' + \beta f(k') \right] \\ &= \theta \sup_{k' \in \Gamma(k)} \left[\tilde{A}k - k' + \beta f(k') \right] \\ &= \theta(Tf)(k) \end{aligned}$$

since $\theta k' \in \Gamma(\theta k) \iff k' \in \Gamma(k)$ by part 1.

9. Show that T satisfies Blackwell's sufficient conditions for a contraction (and hence there exists a unique fixed point of (FE) $v = Tv$).

6.7 Appendix - Proofs for Chapter 6

Proof of Dini's Theorem 453. Let $\langle f_n \rangle$ be decreasing, $f_n \rightarrow f$ pointwise, and define $\bar{f}_n = f_n - f$. Then $\langle \bar{f}_n \rangle$ is a decreasing sequence of non-negative functions with $\bar{f}_n \rightarrow 0$ pointwise. For a given $x \in X$ and $\varepsilon > 0$, $\exists N(\varepsilon, x)$ such that $0 \leq \bar{f}_{N(\varepsilon, x)}(x) < \varepsilon$. Since $\bar{f}_{N(\varepsilon, x)}$ is continuous $\exists \delta(x)$ such that $0 \leq \bar{f}_{N(\varepsilon, x)}(x') < \varepsilon$ for all $x' \in B_{\delta(x)}(x)$. Since \bar{f}_n is decreasing, $0 \leq \bar{f}_n(x') < \varepsilon$ for all $n \geq N(\varepsilon, x)$ we have $x' \in B_{\delta(x)}(x)$. Since the collection $\{B_{\delta(x)}(x), x \in X\}$ is an open covering of X , there exists a finite subcovering of X (i.e. $X = \cup_{i=1}^k B_{\delta(x_i)}(x_i)$). Define $N(\varepsilon) = \min_{i=1, \dots, k} \{N(\varepsilon, x_i)\}$ which is well defined since $N(\varepsilon)$ is just the minimum of a finite set. For a given ε , we found $N(\varepsilon)$ such that $0 \leq \bar{f}_n(x) < \varepsilon$ for all $n \geq N(\varepsilon)$ and for all $x \in X$ (i.e. $\bar{f}_n \rightarrow 0$ uniformly so that $f_n \rightarrow f$). ■

Proof of Lemma 456. (\Leftarrow) This direction is apparent. (\Rightarrow) Let $\mathcal{D} \subset C(X)$ be equicontinuous. Then given $x \in X$ and $\varepsilon > 0$, $\exists \delta(\varepsilon, x)$ such that $|h(x') - h(x)| < \frac{\varepsilon}{2}$ for all x' such that $d_X(x, x') < \delta$ for all $h \in \mathcal{D}$. The collection of open balls $\left\{ B_{\frac{1}{2}\delta(\varepsilon, \frac{\varepsilon}{2})}(x), x \in X \right\}$ is an open covering of X and since X is compact there exists finitely many x_1, \dots, x_k s.t. $\left\{ B_{\frac{1}{2}\delta(x_i, \frac{\varepsilon}{2})}(x_i), i = 1, \dots, k \right\}$ covers X . Let $\delta \equiv \frac{1}{2} \min \left\{ \delta(x_i, \frac{\varepsilon}{2}), i = 1, \dots, k \right\}$. For $x \in X$, then $\exists i$ such that $x \in B_{\delta(x_i, \frac{\varepsilon}{2})}(x_i)$. Let $y \in X$ such that $d_X(x, y) \leq$

δ . Then $d_X(y, x_i) \leq d_X(y, x) + d_X(x, x_i) \leq \delta + \frac{1}{2}\delta \left(x_i, \frac{\varepsilon}{2}\right) \leq \delta \left(x_i, \frac{\varepsilon}{2}\right)$. Therefore for any $h \in \mathcal{D}$, $|h(x) - h(y)| \leq |h(x) - h(x_i)| + |h(x_i) - h(y)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ since \mathcal{D} is equicontinuous at x_i . Hence \mathcal{D} is uniformly equicontinuous. ■

Proof of Lemma 457. (\Leftarrow) Suppose that \mathcal{D} is totally bounded. Let $\varepsilon > 0$ be given and choose positive numbers ε_1 and ε_2 such that $2\varepsilon_1 + \varepsilon_2 \leq \varepsilon$. Total boundedness of \mathcal{D} implies that there exist finitely many functions f_1, \dots, f_n such that the collection of open balls $\{B_{\varepsilon_1}(f_i), i = 1, \dots, n\}$ covers \mathcal{D} . Fix x_0 . Because $\{f_i, i = 1, \dots, n\}$ is equicontinuous at x_0 (since a finite subset of continuous functions is equicontinuous), there exists $\delta > 0$ such that $d_Y(f_i(x), f_i(x_0)) < \varepsilon_2$ for all x such that $d(x, x_0) < \delta$ and for all $i = 1, \dots, n$. To prove that \mathcal{D} is equicontinuous at x_0 we need to show that $d_Y(f(x), f(x_0)) < \varepsilon$ for all x such that $d_X(x, x_0) < \delta$ and all $f \in \mathcal{D}$. Let $f \in \mathcal{D}$. Because $\{B_{\varepsilon_1}(f_i), i = 1, \dots, n\}$ covers \mathcal{D} , then $\exists f_i$ such that $f \in B_{\varepsilon_1}(f_i)$. By the triangle inequality

$$\begin{aligned} d_Y(f(x), f(x_0)) &\leq d_Y(f(x), f_i(x)) + d_Y(f_i(x), f_i(x_0)) + d_Y(f_i(x_0), f(x_0)) \\ &\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_1 \leq \varepsilon \end{aligned}$$

holds true for all x such that $d_X(x, x_0) < \delta$ and for all $f \in \mathcal{D}$. Notice that this direction doesn't require compactness of either X nor Y , hence total boundedness always implies equicontinuity.

(\Rightarrow) Suppose \mathcal{D} is equicontinuous. Given $\varepsilon > 0$ we wish to cover \mathcal{D} by finitely many open ε -balls. Choose ε_1 and ε_2 such that $2\varepsilon_1 + \varepsilon_2 \leq \varepsilon$. Using equicontinuity of \mathcal{D} at $x \in X$, given ε_1 , $\exists \delta(\varepsilon_1, x)$ such that for $x' \in B_{\delta(\varepsilon_1, x)}(x)$, $d(f(x'), f(x)) < \varepsilon_1$ for all $f \in \mathcal{D}$. The collection $\{B_{\delta(\varepsilon_1, x)}(x), x \in X\}$ is an open covering of X . Since X is compact there exist finitely many x_1, \dots, x_k such that $\{B_{\delta(\varepsilon_1, x_i)}(x_i), i = 1, \dots, k\}$ covers X and $d_Y(f(x), f(x_i)) < \varepsilon_1$ holds for $x \in B_{\delta(\varepsilon_1, x_i)}(x_i)$ and all $f \in \mathcal{D}$. Now cover Y by finitely many open balls $\{B_{\varepsilon_2}(y_j), j = 1, \dots, m\}$. Let J be the set of all functions $\alpha : \{1, \dots, k\} \rightarrow \{1, \dots, m\}$. The set J is finite. Given $\alpha \in J$, if there exists a function $f \in \mathcal{D}$ such that $f(x_i) \in B_{\varepsilon_2}(y_{\alpha(i)})$ for each $i = 1, \dots, k$, choose one such function and label it f_α . The finite collection of open balls $\{B_\varepsilon(f_\alpha), \alpha \in J\}$ with $\varepsilon \leq 2\varepsilon_1 + \varepsilon_2$ covers \mathcal{D} . For each $i = 1, \dots, k$, choose an integer $\alpha(i)$ such that $f(x_i) \in B_{\varepsilon_2}(y_{\alpha(i)})$. For this index α , the ε -ball around f_α contains f (i.e. $f \in B_\varepsilon(f_\alpha)$). Let $f \in \mathcal{D}$. Then $f(x_i) \in B_{\varepsilon_2}(y_{j(i)})$ for $i = 1, \dots, k$ because $\{B_{\varepsilon_2}(y_j), j = 1, \dots, m\}$ covers all of Y (which is possible since Y is compact and thus totally bounded). Define

the function $f_{j(i)}$ such that $f_{j(i)}(x_i) \in B_{\varepsilon_2}(y_{j(i)})$ for $i = 1, \dots, k$. Let $x \in X$. Choose i such that $x \in B_{\delta(\varepsilon_1, x_i)}(x_i)$. Then

$$\begin{aligned} d_Y(f(x), f_{j(i)}(x)) &\leq d(f(x), f(x_i)) + d(f(x_i), f_\alpha(x_i)) + d(f_\alpha(x_i), f_\alpha(x)) \\ &\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_1 < \varepsilon. \end{aligned}$$

■

Proof of Lemma 467. By construction. Define $P_n(x)$ on $[-1, 1]$ by induction: $P_1(x) = 0$ and $P_{n+1}(x) = P_n(x) + \frac{1}{2}(x^2 - P_n^2(x))$, $\forall n \in \mathbb{N}$.

To prove that $P_n(x) \rightarrow |x|$ on $[-1, 1]$ uniformly, we will use Dini's Lemma 453. In that case we must check: (i) $P_n(x) \leq P_{n+1}(x)$, $\forall x \in [-1, 1]$; and (ii) $P_n(x)$ converges to $|x|$ pointwise on $[-1, 1]$. We check that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$, $\forall x \in [-1, 1]$ by induction.

To show $\langle P_n \rangle$ is non-decreasing, suppose it holds for $n \geq 1$ ($n = 1$ is clear). Then $P_{n+2}(x) = P_{n+1}(x) + \frac{1}{2}(x^2 - P_{n+1}^2(x)) \geq P_{n+1}(x)$ because $0 \leq P_{n+1}(x) \leq |x| \Leftrightarrow P_{n+1}^2(x) \leq |x|^2$.

To show $P_{n+2} \leq |x|$, use the identity

$$P_{n+2} = |x| - (|x| - P_{n+1}(x)) \left(1 - \frac{1}{2}[|x| + P_{n+1}(x)]\right).$$

Since $|x| - P_{n+1}(x) \geq 0$ by assumption, $|x| + P_{n+1}(x) \leq 2|x|$ and hence $1 - \frac{1}{2}[|x| + P_{n+1}(x)] \geq 0$.

Thus the sequence $\langle P_n(x) \rangle$ is increasing and bounded $\forall x \in [-1, 1]$ and therefore it converges to a function $f(x)$. Taking the limit of $P_{n+1}(x) = P_n(x) + \frac{1}{2}(x^2 - P_n^2(x))$ yields $f = f + \frac{1}{2}(x^2 - f^2)$ which implies $f^2(x) = x^2$ or $f(x) = |x|$ (which we know is continuous). By Dini's lemma 453 $\langle P_n(x) \rangle$ converges to $|x|$ uniformly on $[-1, 1]$. ■

Proof of Schauder's Fixed Point Theorem 475.

Since K is compact, K is totally bounded. Hence, given any $\varepsilon > 0$, there exists a finite set $\{y_i, i = 1, \dots, n\}$ such that the collection $\{B_\varepsilon(y_i), i = 1, \dots, n\}$ covers K . We now define the convex hull $K_\varepsilon = \{\theta_1 y_1 + \dots + \theta_n y_n : \sum \theta_i = 1, \text{ all } \theta_i \geq 0\}$. This is a subset of K since K is convex and K contains all the points y_i . We will now map all of K into K_ε by a continuous function $P_\varepsilon(y)$ that approximates y (i.e. $\|P_\varepsilon(y) - y\| < \varepsilon, \forall y \in K$). To construct this function $P_\varepsilon(y)$, we must construct n continuous functions $\theta_i = \theta_i(y) \geq 0$, with $\sum_{i=1}^n \theta_i = 1$. First, for $i = 1, \dots, n$, we define

$$\varphi_i(y) = \begin{cases} 0 & \text{if } |y_i - y| \geq \varepsilon \\ \varepsilon - |y_i - y| & \text{if } |y_i - y| < \varepsilon \end{cases}, i = 1, \dots, n.$$

Each of these n functions $\varphi_i(y)$ is continuous and the fact that the set $\{y_1, \dots, y_n\}$ is dense in K guarantees $\varphi_i(y) > 0$ for some $i = 1, \dots, n$. Now construct continuous functions

$$\theta_i(y) = \frac{\varphi_i(y)}{\sum_{i=1}^n \varphi_i(y)}, i = 1, \dots, n, y \in K.$$

These functions are well-defined since $\sum_{i=1}^n \varphi_i(y) > 0$. The functions $\theta_i(y)$ satisfy $\theta_i \geq 0$, $\sum \theta_i = 1$. Finally we construct the continuous function

$$P_\varepsilon(y) = \theta_1(y)y_1 + \dots + \theta_n(y)y_n.$$

This function maps K into K_ε . From the construction of φ_i , $\theta_i(y) = 0$ unless $\|y_i - y\| < \varepsilon$. Therefore $P_\varepsilon(y)$ is a convex combination of just those points y_i for which $\|y_i - y\| < \varepsilon$. Hence

$$\|P_\varepsilon(y) - y\| = \left\| \sum \theta_i(y)y_i - y \right\| = \left\| \sum \theta_i(y)(y_i - y) \right\| \leq \sum \theta_i(y)\|y_i - y\| < \varepsilon.$$

This establishes that $P_\varepsilon(y)$ approximates y . Now we map the convex set K_ε continuously into itself by the function $f_\varepsilon : K_\varepsilon \rightarrow K_\varepsilon$ where $f_\varepsilon(x) \equiv P_\varepsilon(f(x))$ for all $x \in K_\varepsilon$. Since K_ε is a convex, compact, finite-dimensional vector subspace spanned by the n points y_1, \dots, y_n and $f_\varepsilon : K_\varepsilon \rightarrow K_\varepsilon$ is continuous, there exists a fixed point $x_\varepsilon = f_\varepsilon(x_\varepsilon)$ in K_ε due to Brouwer's fixed point theorem 302. Now we take the limit as $\varepsilon \rightarrow 0$. Set $y_\varepsilon = f(x_\varepsilon)$. Since K is compact, we may let $\varepsilon \rightarrow 0$ through some sequence $\varepsilon_1, \varepsilon_2, \dots$ for which $\langle y_\varepsilon \rangle$ converges to a limit in K :

$$f(x_\varepsilon) = y_\varepsilon \rightarrow \bar{y} \text{ as } \varepsilon_k \rightarrow 0. \quad (6.32)$$

We now write

$$\begin{aligned} x_\varepsilon &= f_\varepsilon(x_\varepsilon) \equiv P_\varepsilon(f(x_\varepsilon)) = P_\varepsilon(y_\varepsilon) \\ x_\varepsilon &= y_\varepsilon + [P_\varepsilon(y_\varepsilon) - y_\varepsilon] \end{aligned}$$

Then

$$\|x_\varepsilon - \bar{y}\| = \|y_\varepsilon + P_\varepsilon(y_\varepsilon) - y_\varepsilon - \bar{y}\| = \|P_\varepsilon(y_\varepsilon) - \bar{y}\| \leq \|P_\varepsilon(y_\varepsilon) - y_\varepsilon\| + \|y_\varepsilon - \bar{y}\|.$$

The first term vanishes since $P_\varepsilon(y)$ approximates y and the second term vanishes since y_ε converges to \bar{y} as $\varepsilon_k \rightarrow 0$. Hence $x_\varepsilon \rightarrow \bar{y}$ as $\varepsilon = \varepsilon_k \rightarrow 0$.

Now since f is continuous then $f(x_\varepsilon) \rightarrow f(\bar{y})$. Combining this and (??) we have $f(\bar{y}) = \bar{y}$, for some $\bar{y} \in K$. Hence \bar{y} is a fixed point of f . ■

Proof of Riesz-Fischer Theorem 481. The case for $p = \infty$ is in the text.

Second, let $p \in [1, \infty)$. Let $\langle f_n \rangle$ be a Cauchy sequence in L^p . In order to find a function to which the sequence converges in light of Example 444, we need to take a more sophisticated approach than for $p = \infty$. Since $\langle f_n \rangle$ is Cauchy we can recursively construct a strictly increasing sequence $\langle n_j \rangle$ in \mathbb{N} such that $\|f_k - f_n\|_p < 2^{-j}$, $\forall k, n \geq n_j$ and $\forall j \in \mathbb{N}$. Then

$$\left\| \sum_{j=1}^J |f_{n_{j+1}} - f_{n_j}| \right\|_p \leq \sum_{j=1}^J \|f_{n_{j+1}} - f_{n_j}\|_p < \sum_{j=1}^J 2^{-j} < 1$$

for each $J \in \mathbb{N}$ by the Minkowski inequality in Theorem 480. Therefore,

$$\begin{aligned} \int_X \left(\sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| \right)^p &= \int_X \left(\lim_{J \rightarrow \infty} \sum_{j=1}^J |f_{n_{j+1}} - f_{n_j}| \right)^p \\ &= \int_X \lim_{J \rightarrow \infty} \left(\sum_{j=1}^J |f_{n_{j+1}} - f_{n_j}| \right)^p \\ &= \lim_{J \rightarrow \infty} \int_X \left(\sum_{j=1}^J |f_{n_{j+1}} - f_{n_j}| \right)^p \\ &= \lim_{J \rightarrow \infty} \left(\left\| \sum_{j=1}^J |f_{n_{j+1}} - f_{n_j}| \right\|_p \right)^p \leq 1 \end{aligned}$$

where the third equality follows from the Monotone Convergence Theorem 396. This implies that the sum $\sum_{j=1}^J |f_{n_{j+1}} - f_{n_j}|$ is finite a.e. This means there exists a set A such that $mA = 0$ and $\sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}|$ converges on $X \setminus A$. Since $f_{n_{j+1}} - f_{n_j} \leq |f_{n_{j+1}} - f_{n_j}|$, we have $\sum_{j=1}^{\infty} f_{n_{j+1}} - f_{n_j}$ converges on $X \setminus A$. Let $f(x)$, $x \in X \setminus A$ be the limit of this series. Then by Theorem 364 (the pointwise limit of measurable functions is measurable), f is measurable.

Finally, we need to show that f is also p -integrable. To do so, suppose that $\varepsilon > 0$ is given and let N_0 be an integer such that $\|f_k - f_n\|_p < \varepsilon$ for

$k, n \geq N_0$. Then

$$\begin{aligned} \|f_n - f\|_p &= \left(\int_X |f_n - f|^p \right)^{\frac{1}{p}} = \left(\int_X \lim_{j \rightarrow \infty} |f_n - f_{n_j}|^p \right)^{\frac{1}{p}} \\ &\leq \left(\liminf_{j \rightarrow \infty} \int_X |f_n - f_{n_j}|^p \right)^{\frac{1}{p}} \leq \varepsilon, \forall n \geq N_0 \end{aligned}$$

where the second equality follows since $\langle f_n \rangle$ is Cauchy and the first inequality follows by Fatou's Lemma 393. Since $\|f\|_p = \|f - f_n + f_n\|_p \leq \|f - f_n\|_p + \|f_n\|_p \leq \varepsilon + \|f\|_p < \infty$, then $f \in L_p$ and $f_n \rightarrow f$ in L_p . ■

Proof of Theorem 520. Let $\langle T_n \rangle$ be a Cauchy sequence in $\mathfrak{BL}(X, Y)$. For a fixed $x \in X$ we have

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\| \|x\|_X$$

so that $\langle T_n(x) \rangle$ is a Cauchy sequence in Y . Since Y is complete, $\langle T_n(x) \rangle$ converges to an element $y \in Y$. Call this element Tx ($Tx = \lim_{n \rightarrow \infty} T_n(x)$, $\forall x \in X$). Thus we can define $T : X \rightarrow Y$ by $Tx = \lim_{n \rightarrow \infty} T_n(x)$. We must show that T is bounded and that $\langle T_n \rangle \rightarrow T$ as $n \rightarrow \infty$.

Since $\langle T_n \rangle$ is Cauchy, then given $\varepsilon > 0 \exists N$ such that $\forall m, n \geq N$ we have $\|T_n - T_m\| < \varepsilon$. Hence $\|T_n - T_N\| < \varepsilon, \forall n \geq N$ or $\|T_n\| < \|T_N\| + \varepsilon, \forall n \geq N$. Thus, $\|Tx\|_Y = \lim_{n \rightarrow \infty} \|T_n x\|_Y \leq \lim_{n \rightarrow \infty} (\|T_n\| \|x\|_X) \leq (\|T_N\| + \varepsilon) \|x\|_X$. Thus T is bounded.

For each $x \in X$ we have $\|T_n x - Tx\|_Y = \lim_{m \rightarrow \infty} \|T_n x - T_m x\|_Y \leq \lim_{m \rightarrow \infty} \|T_n - T_m\| \|x\|_X \leq \varepsilon \|x\|_X, \forall n \geq N$ where the inequality follows from Corollary 519. Thus $\|T_n - T\| = \sup \{ \|(T_n - T)x\|_Y, \|x\|_X = 1 \} \leq \varepsilon \|1\| = \varepsilon$. Thus $T_n \rightarrow T$ in $\mathfrak{BL}(X, Y)$. ■

Proof of Theorem 529. Let $G : X \rightarrow \mathbb{R}$ be any bounded linear functional on \mathbb{R}^n from X^* (i.e. $G \in X^*$). Let $\{e^1, \dots, e^n\}$ be the natural basis in \mathbb{R}^n ¹² and define $b_i = G(e^i)$ for $i = 1, \dots, n$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we have

$$\begin{aligned} G(x) &= G(x_1 e^1 + \dots + x_n e^n) \\ &= x_1 G(e^1) + \dots + x_n G(e^n) \\ &= x_1 b_1 + \dots + x_n b_n \\ &= \langle x, b \rangle. \end{aligned}$$

¹²Recall that the natural (or canonical) basis in \mathbb{R}^n is defined to be the set of vectors $\{e_1, \dots, e_n\}$ where $e_i = (0, \dots, 1, \dots, 0)$ with '1' in the i th place.

Clearly the functional G is represented by the point $b = (b_1, \dots, b_n) \in \mathbb{R}^n$. Next we show that $\|G\| = \|b\|_X$. First,

$$|G(x)| \leq \sum_{i=1}^n |x_i G(e^i)| \leq \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n G(e^i)^2 \right)^{\frac{1}{2}} = \|x\|_X \|b\|_X$$

where the first inequality follows from the triangle inequality and the second inequality follows from the Cauchy-Schwartz inequality (Theorem 210). Hence $\|G\| \leq \|b\|_X$ by (iv) of Theorem 518. Second, choose $x_0 = (b_1, \dots, b_n)$. Then we have

$$\|G\| \geq \frac{|G(x_0)|}{\|x_0\|_X} = \frac{|G(b)|}{\|b\|_X} = \frac{\langle b, b \rangle}{\|b\|_X} = \frac{\|b\|_X^2}{\|b\|_X} = \|b\|_X$$

where the inequality follows from Corollary 519. Combining these two inequalities we have $\|G\| = \|b\|_X$.

It is easy to show that each functional $G \in X^*$ is *uniquely* represented by the point $b \in \mathbb{R}^n$. To see this, it is sufficient to prove that an operator $T : X^* \rightarrow X$ defined by $T(G) = (G(e^1), \dots, G(e^n)) = b$ is a bounded, linear bijection such that $G(x) = \langle b, x \rangle$. To see that T is bounded (and hence continuous by Theorem 511) note

$$\|T\| = \sup \left\{ \frac{\|TG\|_X}{\|G\|_{X^*}}, \|G\|_{X^*} \neq 0 \right\} = \sup \left\{ \frac{\|b\|_X}{\|b\|_X}, \|b\|_X \neq 0 \right\} = 1.$$

To see that T is linear, since $\langle (\alpha b + \beta b'), x \rangle = \alpha \langle b, x \rangle + \beta \langle b', x \rangle$, we know $T(\alpha b + \beta b') = \alpha T(b) + \beta T(b')$. To see that T is a bijection, first we establish it is an injection (i.e. one-to-one). Let $G_1 \neq G_2$. Then $\exists x \in X$ such that $G_1(x) \neq G_2(x)$. Since $x = x_1 e^1 + \dots + x_n e^n$ uniquely, we have

$$\begin{aligned} G_1(x) &= G_1(x_1 e^1 + \dots + x_n e^n) \\ &= x_1 G_1(e^1) + \dots + x_n G_n(e^n) \\ &= x_1 b_1^1 + \dots + x_n b_n^1 \end{aligned}$$

and similarly

$$G_2(x) = x_1 b_1^2 + \dots + x_n b_n^2.$$

Then $G_1(x) \neq G_2(x) \Rightarrow b_1^1 \neq b_1^2$ or \dots $b_n^1 \neq b_n^2$ so that $TG_1 \neq TG_2$. To see T is a surjection (onto), we must show that for any $d \in X$, $\exists G \in X^*$ such that $T(G) = d$. But $G(x) = \langle d, x \rangle \in X^*$ and $TG = d$.¹³ ■

¹³For example, if $X = \mathbb{R}^2$ and we take $d = (3, 4) \in X$, then $G_{(3,4)}(x) = 3x_1 + 4x_2 \in X^*$.

Proof of Theorem 530. Let $\{e^i, i \in \mathbb{N}\}$ is a complete orthonormal system in \mathcal{H} . Set $b_i = F(e^i), \forall i \in \mathbb{N}$. Then we have $\sum_{i=1}^n b_i^2 = F(\sum_{i=1}^n b_i e^i) \leq$

$\|F\| \|\sum_{i=1}^n b_i e^i\| = \|F\| \left(\sum_{i=1}^n b_i^2\right)^{\frac{1}{2}}$. Taking the square of both sides, we have

$\sum_{i=1}^n b_i^2 \leq \|F\|^2$ for arbitrary n . Then $\sum_{i=1}^{\infty} b_i^2 \leq \|F\|^2 < \infty$ which means

that the series $\sum_{i=1}^{\infty} b_i^2$ is convergent. Then there exists an element $y \in$

\mathcal{H} whose Fourier coefficients are $b_i, i \in \mathbb{N}$. Since $\{e^i, i \in \mathbb{N}\}$ is a complete orthonormal system (by Parseval's equality) we have $b = \sum_{i=1}^{\infty} b_i e^i$

and also $\|b\| \leq \|F\|$. Let x be any element of \mathcal{H} and let $\{x_i; i \in \mathbb{N}\}$ be its Fourier coefficients. Then $\sum_{i=1}^n x_i e^i \rightarrow x$ by Parseval's Theorem 504.

Since F is linear, $F(x) = \lim_{n \rightarrow \infty} F\left(\sum_{i=1}^n x_i e^i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i F(e^i) =$

$\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i b_i = \sum_{i=1}^{\infty} x_i b_i = \langle x, b \rangle$. By the Cauchy-Schwarz inequality

$|F(x)| \leq \|x\| \|b\|, \forall x \in \mathcal{H}$, so that $\|F\| \leq \|b\|$. Combining the two inequalities, we have $\|F\| = \|b\|$. ■

Proof of Theorem 531. Suppose $x = (x_1, \dots, x_n, \dots) \in \ell_p$ and $F \in \ell_p^*$. Set $\{e^i, i \in \mathbb{N}\}$ where e^i is the vector having the i -th entry equal to one and all other entries equal to zero.

Let $s_n = \sum_{i=1}^n x_i e^i$. Then $s_n \in \ell_p$ and $\|x - s_n\|_p^p = \sum_{i=n+1}^{\infty} |x_i|^p \rightarrow 0$ as $n \rightarrow \infty$. Thus $F(s_n) = F(\sum_{i=1}^n x_i e^i) = \sum_{i=1}^n x_i F(e^i)$ and $|F(x) - F(s_n)| = |F(x - s_n)| \leq \|F\| \|x - s_n\|_p \rightarrow 0$ as $n \rightarrow \infty$. Hence $F(x) = \lim_{n \rightarrow \infty} F(s_n) = \sum_{i=1}^{\infty} x_i F(e^i)$. Set $z_i = F(e^i), i \in \mathbb{N}$ and $z = \langle z_1, \dots, z_n, \dots \rangle$. We must show that $z \in \ell_q$. For this purpose choose a particular $x = \langle x_i \rangle$:

$$x_i = \begin{cases} |z_i|^{q-2} z_i & \text{when } z_i \neq 0 \\ 0 & \text{when } z_i = 0 \end{cases}.$$

For this case $\|s_n\|_p^p = \sum_{i=1}^n |x_i|^p = \sum_{i=1}^n |z_i|^{p(q-1)} = \sum_{i=1}^n |z_i|^2$. Moreover

$F(s_n) = \sum_{i=1}^n x_i z_i = \sum_{i=1}^n |z_i|^q$ and $|F(s_n)| \leq \|F\| \|s_n\|_p \leq \|F\| (\sum_{i=1}^n |z_i|^q)^{\frac{1}{p}}$. Hence $\sum_{i=1}^n |z_i|^q \leq \|F\| (\sum_{i=1}^n |z_i|^q)^{\frac{1}{p}} \Leftrightarrow (\sum_{i=1}^n |z_i|^q)^{\frac{1}{q}} \leq \|F\|$ holds true for arbitrary n . Thus $z = \langle z_i \rangle \in \ell_q$ and $\|z\|_q \leq \|F\|$. On the other hand by Hölder's inequality we have $|F(x)| = |\sum_{i=1}^{\infty} x_i z_i| \leq \|x\|_p \|z\|_q$ and thus $\|F\| \leq \|z\|_q$. This shows that $\|F\| = \|z\|_q$. ■

Proof of Riesz Representation Theorem 532. Let us first consider $m(X) < \infty$. In that case $\chi_E \in L_p(X)$ for any $E \subset X$ which is \mathcal{L} -measurable (i.e. $E \in \mathcal{L}$). Then defined a set function $\nu : \mathcal{L} \rightarrow \mathbb{R}$ by $\nu(E) = F(\chi_E)$ for $E \subset \mathcal{L}$. ν is a finite signed measure which is absolutely continuous with respect to m . (Show it???) Then by the Radon Nikodym Theorem 434 there is an integrable function g such that $\nu(E) = \int_E g dm$ for all $E \in \mathcal{L}$. Thus we take $F(\chi_E) = \int_E g dm$ for all $E \in \mathcal{L}$. If φ is a simple function (i.e. it is a finite linear combination of characteristic functions), then by linearity of F we have

$$\begin{aligned} F(\varphi) &= F\left(\sum_{i=1}^n c_i \chi_{E_i}\right) = \sum_{i=1}^n c_i F(\chi_{E_i}) = \sum_{i=1}^n c_i \int_{E_i} g dm \\ &= \sum_{i=1}^n c_i \int_X \chi_{E_i} g dm = \int_X \sum_{i=1}^n c_i \chi_{E_i} g dm = \int_X g \varphi dm. \end{aligned}$$

Since $|F(\varphi)| \leq \|F\| \|\varphi\|_p$ we have that $g \in L_q(X)$. (Show it???) Hence there is a function $g \in L_q(X)$ such that $F(\varphi) = \int g \varphi dm$ for all simple functions φ . Since the subset of all simple functions is dense in $L_p(X)$, then $F(f) = \int_X g f dm$ for all $f \in L_p(X)$. (Show it???) Also show that $\|F\| = \|g\|_q$. The function g is determined uniquely for if g_1 and g_2 determine the same functional F , then $g_1 - g_2$ must determine the zero functional; hence $\|g_1 - g_2\|_q = 0$ which implies $g_1 = g_2$ a.e.

Let $m(X) = \infty$. Since m is σ -finite, there is an increasing sequence of \mathcal{L} -measurable sets $\langle X_n \rangle$ with finite measure whose union is X . By the first part of the proof for each n there is a function $g_n \in L_q$ such that g_n vanishes outside X_n and $F(f) = \int f g_n dm$ for all $f \in L_p$ that vanish outside x . Moreover $\|g_n\|_q \leq \|F\|$. Since any function g_n is unique on X_n (except on sets of measure zero), $g_{n+1} = g_n$ on X_n . Set $g(x) = g_n(x)$ for $x \in X_n$. Then g is a well defined \mathcal{L} -measurable function and $|g_n|$ increases pointwise to $|g|$. Thus by the Monotone Convergence Theorem 396

$$\int |g|^q dm = \lim_{n \rightarrow \infty} \int |g_n|^q dm \leq \|F\|^q$$

for $g \in L_q$. For $f \in L_p$ define

$$f_n = \begin{cases} f(x) & \text{for } x \in X_n \\ 0 & \text{for } x \in X \setminus X_n \end{cases}.$$

Then $f_n \rightarrow f$ pointwise and in L_p . By the Holder Inequality 479 $|fg|$ is integrable and $|f_n g| \leq |fg|$ so that by the Lebesgue Dominated Convergence Theorem 404

$$\int f g dm = \lim_{n \rightarrow \infty} \int f_n g dm = \lim_{n \rightarrow \infty} \int f_n g_n dm = \lim_{n \rightarrow \infty} F(f_n) = F(f).$$

■

Proof of Hahn-Banach Theorem 539. If $M = X$, then there is nothing to prove. Thus assume $M \subsetneq X$. Then $\exists x_1 \in X$ which is not in M . Let

$$M_1 = \{w \in X : w = \alpha x_1 + x, \alpha \in \mathbb{R}, x \in M\} \quad (6.33)$$

One can prove (see Exercise 6.7.1 below) that M_1 defined in (6.33) is a subspace of X and that the representation in (6.33) is unique.

Next, extend f to M_1 and call this extension F . In order for $F : M_1 \rightarrow \mathbb{R}$ to be a linear functional, it must satisfy

$$F(\alpha x_1 + x) = \alpha F(x_1) + F(x) = \alpha F(x_1) + f(x) \quad (6.34)$$

where the second equality follows since $x \in M$. Hence F is completely determined by the choice of $F(x_1)$. Moreover, we must have $\alpha F(x_1) + f(x) \leq P(\alpha x_1 + x)$ for all scalars α and $x \in X$. If $\alpha > 0$, this means

$$F(x_1) \leq \frac{1}{\alpha} [p(\alpha x_1 + x) - f(x)] = p\left(x_1 + \frac{x}{\alpha}\right) - f\left(\frac{x}{\alpha}\right) = p(x_1 + z) - p(z)$$

where $z = \frac{x}{\alpha}$. If $\alpha < 0$, we have

$$F(x_1) \geq \frac{1}{\alpha} [p(\alpha x_1 + x) - f(x)] = f(y) - p(-x, +y)$$

where $y = -\frac{x}{\alpha}$. Combining these two inequalities we have

$$f(y) - p(y - x_1) \leq F(x_1) \leq p(x_1 + z) - f(z), \forall y, z \in M \text{ or } M_1 \quad (6.35)$$

Conversely, if we can pick $F(x_1)$ to satisfy (6.35) then will satisfy (6.34) and F will satisfy (6.10) on M_1 . For if $F(x_1)$ satisfies (6.35) then for $\alpha > 0$ we have

$$\alpha F(x_1) + f(x) = \alpha \left[F(x_1) + f\left(\frac{x}{\alpha}\right) \right] \leq \alpha p\left(x_1 + \frac{x}{\alpha}\right) = p(\alpha x_1 + x)$$

while for $\alpha < 0$ we have

$$\alpha F(x_1) + f(x) = -\alpha \left[-F(x_1) + f\left(-\frac{x}{\alpha}\right) \right] \leq -\alpha p\left(-\frac{x}{\alpha} - x_1\right) = p(\alpha x_1 + x).$$

So we have now reduced the problem to finding a value $F(x_1)$ to satisfy (6.35). In order for such a value to exist, we must have

$$f(y) - p(y - x_1) \leq p(x_1 + z) - f(z), \forall y, z \in M \quad (6.36)$$

In other words we need

$$f(y + z) \leq p(x_1 + z) + p(y - x_1)$$

But this is true by (6.8). Hence (6.36) holds. If we fix y and let z run through all elements of M , we have

$$f(y) - p(y - x_1) \leq \inf_{z \in M} \{p(x_1 + z) - f(z)\} \equiv C.$$

Since this is true for any $y \in M$, we have

$$c \equiv \sup_{y \in M} \{f(y) - p(y - x_1)\} \leq C.$$

We now pick $F(x_1)$ to satisfy $c \leq F(x_1) \leq C$. Note that the extension is unique only when $c = C$. Thus we have extended f from M to M_1 . If $M_1 = X$ we are done. Otherwise, there is an element $x_2 \in X$ not in M_1 . Let M_2 be the space spanned by M_1 and x_2 ($M_2 = \alpha x_2 + x, \alpha \in \mathbb{R}, x \in M_1$). By repeating the process we can extend f to M_2 .

If we prove that the collection of all linear bounded functionals defined on subspaces of X satisfies the assumptions of Zorn's lemma we are done (because Zorn's lemma guarantees the existence of a maximal element which we will prove is the desired functional). Consider the collection L of all linear functionals $g : D(g) \rightarrow \mathbb{R}$ defined on a vector subspace of X such that the vector subspace satisfies: (i) $D(g) \supset M$; (ii) $g(x) = f(x), \forall x \in M$; (iii) $g(x) \leq p(x), \forall x \in D(g)$. Note that L is not empty since F belongs there. Introduce a partial ordering " \prec " in L as follows. If $D(g_1) \subset D(g_2)$ and $g_1(x) = g_2(x) \forall x \in D(g_1)$, then $g_1 \prec g_2$. One can prove (see Exercise 6.7.2 below) that " \prec " defined above is a partial ordering in L .

We have to check now that every totally ordered subset of L has an upper bound in L . Let W be a totally ordered subset of L . Define the functional h by

$$\begin{aligned} D(h) &= \cup_{g \in W} D(g) \\ h(x) &= g(x), g \in W, x \in D(g). \end{aligned}$$

Clearly $h \in L$ and it is an upper bound for W . Note that the definition of h is not ambiguous because if g_1, g_2 are any two elements of W , then either $g_1 \subset g_2$ or $g_2 \subset g_1$ and in either case if $x \in D(g_1) \cap D(g_2)$, then $g_1(x) = g_2(x)$. Hence this shows that the assumptions of Zorn's lemma are met and therefore a maximal element F of L exists.

We must show that F is the desired functional. That means that $D(F) = X$. Suppose by contradiction that $D(F) \subsetneq X$. Then $\exists x_0 \in X \setminus D(F)$ and by repeating the process that we used at the beginning we would construct the extension h of F such that $h \supset F$ and $h \neq F$. This would violate the maximality of F . ■

Exercise 6.7.1 Prove that M_1 defined in (??) is a subspace of X and that the representation in (??) is unique.

Exercise 6.7.2 Prove that the relation " \prec " defined in the proof of Theorem 539 is a partial ordering in L .

Proof of Separation Theorem 549. Suppose without loss of generality that $\mathbf{0}$ is an internal point of K_1 . Then $K_1 - K_2 = \{x - y : x \in K_1, y \in K_2\}$ is convex by Theorem 216. Let $x_0 \in K_2$. Since $\mathbf{0}$ is an internal point of K_1 , then $\mathbf{0} - x_0 = -x_0$ must be internal point of $K_1 - K_2$. Let $K = x_0 + K_1 - K_2$. Then K is convex and $\mathbf{0} \in K$ is its internal point. See Figure 6.5.4.

We claim that x_0 is not an internal point of K . Suppose it was. Then $\mathbf{0}$ would be an internal point of $K_1 - K_2$. Then for any $y \neq \mathbf{0}$ and some positive number α , the point αy would belong to $K_1 - K_2$ (i.e. $\alpha y = k_1 - k_2$ for some $k_1 \in K_1$ and $k_2 \in K_2$). This implies $\frac{\alpha y + k_2}{1 + \alpha} = \frac{k_1}{1 + \alpha}$. If y is a point of K_2 then the left-hand side represents a point in K_2 because $\frac{\alpha}{1 + \alpha}y + \frac{1}{1 + \alpha}k_2$ is a convex combination of two points of a convex set K_2 . Furthermore, the right-hand side is an internal point of K_1 because $k_1 \in K_1$, $\mathbf{0} \in K_1$ and $\frac{1}{1 + \alpha} < 1$. This contradicts the assumption that K_2 contains no internal points of K_1 . Thus if $P(x)$ is the support function of K and x_0 is not an internal point of K we know by (iii) of Lemma 546 that $P(x_0) \geq 1$.

Let M be a one-dimensional linear subspace spanned by x_0 (i.e. $M = \{x : x = \alpha x_0, \alpha \in \mathbb{R}\}$). Define a linear functional $f : M \rightarrow \mathbb{R}$ by $f(\alpha x_0) = \alpha P(x_0)$. We must check that f satisfies the assumptions of the Hahn-Banach Theorem 539. Is $f(\alpha x_0) \leq P(\alpha x_0)$ for all α ? If $\alpha \leq 0$, then $f(\alpha x_0) \leq 0$ and hence $f(\alpha x_0) \leq P(\alpha x_0)$ since P is non-negative. If $\alpha > 0$, then $f(\alpha x_0) = \alpha P(x_0) = P(\alpha x_0)$ by property of (i) of P in Lemma 546. Now by the Hahn-Banach Theorem $f(x)$ can be extended to a linear functional $F : X \rightarrow \mathbb{R}$ satisfying $F(x) \leq P(x)$ for all $x \in X$. Thus for $x \in K$ we have $F(x) \leq 1$, $x = x_0 + y - z$ with $y \in K_1$ and $z \in K_2$. Then $x - y + x_0 \in K$ and we have $F(x - y + x_0) \leq 1$ and $F(x) - F(y) + F(x_0) \leq 1$. Since $F(x_0) \geq 1$, we have $F(x) \leq F(y)$ for any $x \in K_1$ and $y \in K_2$. Then we have

$$\sup_{x \in K_1} F(x) \leq \inf_{y \in K_2} F(y)$$

and hence F separates K_1, K_2 and F is a non-zero functional (since $F(x_0) \geq 1$).???Check x, y, z ??? ■

Proof of Second Welfare Theorem 552. Proof. Since S is finite dimensional (A5) and the aggregate technological possibilities set is convex (A4), for the existence of ϕ we must show that the set of allocations preferred to $\{x_i^*\}_{i=1}^I$ given by $A = \sum_{i=1}^I A_i$ is convex where $A_i = \{x \in X_i : u_i(x) \geq u_i(x_i^*), \forall i\}$. Assumptions (A1) – (A3) are sufficient to guarantee that each A_i is convex and so A is convex. Finally, we show that A does not contain any interior points of Y . Suppose to the contrary that $y \in \text{int}Y$ and $y \in A$. Thus, for some $\{x_i\}_{i=1}^I$ with $x_i \in A_i$ for all i , we have $y = \sum_{i=1}^I x_i$. By assumption, there is some $h \in \{1, \dots, I\}$, $\exists \hat{x}_h$ such that $u_h(\hat{x}_h) > u_h(x_h^*)$. Let $x_h^\alpha = \alpha \hat{x}_h + (1 - \alpha)x_h$, $\alpha \in (0, 1)$. By A1 and A2, $x_h^\alpha \in X_h$ and $u_h(x_h^\alpha) > u_h(x_h^*)$. Let $y^\alpha = \sum_{i \neq h} x_i + x_h^\alpha$. Since $y \in \text{int}Y$, it follows that for some sufficiently small ε , $y^\varepsilon \in Y$. In this case the allocation $(\{x_{i \neq h}\}_{i=1}^I \cup x_h^\varepsilon, y^\varepsilon)$ is feasible and satisfies

$$x_i \in X_i, \forall i, u_i(x) \geq u_i(x_i^*), \forall i \neq h, \text{ and } u_h(x_h^\alpha) > u_h(x_h^*)$$

which contradicts the Pareto Optimality of $(\{x_i^*\}_{i=1}^I, \{y_j^*\}_{j=1}^J)$. Therefore the conditions for Theorem 549 are met.

To complete the proof, it is sufficient to show (b) holds in the definition of a competitive equilibrium. By (6.15), suppose that $x_i \in X_i$ and $\phi(x_i) < \phi(x_i^*)$. Hence it follows by contraposition of (6.13) that $u_i(x_i^\alpha) < u_i(x_i^*) \forall \alpha \in (0, 1)$. By A3, $\lim_{\alpha \rightarrow 0} u_i(x_i^\alpha) = u_i(x_i) < u_i(x_i^*)$. ■

6.8 Bibliography for Chapter 6

This material is based on Royden (Chapters) and Munkres (Chapters).

Chapter 7

Topological Spaces

This chapter is a brief overview of topological spaces; it does not go into details nor prove theorems. Let's start with an example about fixed points. In Chapter 4 there is a theorem 257 saying that if $f : I \rightarrow I$ is a continuous mapping from a closed interval I into itself, then there exists a point $x_0 \in I$ such that $f(x_0) = x_0$. Is the theorem still true if the line segment I is distorted (i.e. if it is an arc or an arbitrary curve or a circle)? See Figure 7.1 Since every concept behind the theorem is a topological one, the theorem remains true as long as the object change is homeomorphic.

We will explain the notions of topological properties and homeomorphisms, but at this stage we say that topological properties of an object are those that are invariant with respect to various distortions like bending, increasing (magnifying), decreasing (reducing)-all these transformations are homeomorphic, but are not invariant, for example, to tearing or welding. Thus the theorem remains true for an arc or an arbitrary curve but not for the circle. The first two objects have two ends but the circle does not have any. Thus there is an "inside" and "outside" of the circle but not of the arc or arbitrary curve. It is easy to see that in the case of a circle, the fixed point theorem doesn't hold. Consider a revolution of the circle about an angle-it is a continuous mapping of the circle into itself with no point remaining fixed. See Figure 7.2.????

Definition 585 A set X together with a collection \mathcal{O} (for open sets) which satisfies the following conditions: (i) $\emptyset \in \mathcal{O}$, $X \in \mathcal{O}$; (ii) $(\cup_{i \in \Gamma} A_i) \in \mathcal{O}$, for $A_i \in \mathcal{O}$ (an arbitrary union of elements of \mathcal{O} belongs to \mathcal{O}); (iii) $(\cap_{i=1}^n A_i) \in \mathcal{O}$ for $A_i \in \mathcal{O}$ (a finite intersection of elements of \mathcal{O} belongs to \mathcal{O}). \mathcal{O} is called a

topology on X and its elements are called *open sets*.

Recall the following facts. A set B is called closed if $X \setminus B$ is open. Also \emptyset and X are both open and closed. By using DeMorgan rules we can show that (i) $\cup_{i=1}^n B_i$ is closed for B_i closed and (ii) $\cap_{i \in \Gamma} B_i$ is closed for B_i -closed. The intersection of all closed sets containing a set C is called the closure of C written \bar{C} . Hence the closure of C is the smallest closed set containing C and $C \subset \bar{C}$.

Exercise 7.0.1 Show that C is closed iff $C = \bar{C}$.

The union of all open sets contained in a set D is called the interior of D (written $\text{int}D$) and it is the largest open set contained in D .

Exercise 7.0.2 Show that D is open iff $\text{int}D = D$.

Example 586 Let $(\mathbb{R}, |\cdot|)$ be a metric space. The collection of all open sets (see Def. 104) \mathcal{O} satisfies all three properties of Theorem 106 and hence $|\cdot|$ defines a topology \mathcal{O} in \mathbb{R} . A topology is determined by its metric. You should realize that two equivalent metrics determine the same topology (see notes after the Theorem 221).

Hence any metric space is also a topological space. What about the converse? Consider a topological space X with a topology \mathcal{O} . Does a metric d on X exist that would generate a topology \mathcal{O} .

Definition 587 If there exists a metric d on X that generates a topology \mathcal{O} we say that this topological space is **metrisable**.

Using this definition we can rephrase the question, is any topological space metrisable? The answer is no, as we will see.

Example 588 Given a set X , let \mathcal{O} be the collection of all subsets of X (i.e. \mathcal{O} is the power set of X). This is the largest possible topology on X . We call it the **discrete topology**. The discrete topology is not very interesting. All sets are open (and closed), any mapping from X is continuous. This topological space is metrisable, put $d(x, x) = 0$, and $d(x, y) = 1$ for $x \neq y$ (i.e. the discrete metric).

Example 589 Let X have at least two elements and \mathcal{O} contain only \emptyset and X . This is the smallest possible topological space on X called the **trivial topology** on X . This topological space is not metrisable for the following reason. In any metric space, a set containing just one element is closed. In this topological space the closure of a one element set $\{x\}$ is the whole space X (since this is the only closed set containing x) and hence by Exercise 7.0.1 $\{x\}$ is not closed.

Example 590 Let X be an infinite set. Let \mathcal{O} contain \emptyset and all subsets $A \subset X$ such that $X \setminus A$ is finite. This topology is called the topology of finite complements.

Exercise 7.0.3 Show that \mathcal{O} in the preceding example 590 is a topology on X and that the closure of a set A is

$$\bar{A} = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases}$$

. This topological space is not metrizable as we will see in the next subsection on separation axioms.

Now we will define other topological properties the same way we did in metric spaces. Naturally, we cannot use the notion of distance here; all new objects and properties can be defined only in terms of open sets or in terms of other objects and properties, originally defined by open sets. A neighborhood of a point x is any open set containing x . A point x is a cluster point of a set A if any neighborhood of x contains a point of A different from x .

Exercise 7.0.4 Show that A is closed iff A contains all its cluster points.

As in Definition 153 we say that $S \subset X$ is dense in X if $\bar{S} = X$. In many cases it is rather difficult to define the collection of all the open sets; we often use a small subcollection of open sets to define all open sets. This is exactly the method that we used in a metric space, where open sets were defined in terms of open balls (see Definition 106).

Definition 591 A collection \mathcal{B}_x of open sets is called the **local basis** of X if $x \in B, \forall B \in \mathcal{B}_x$ and if for any neighborhood U of x there exists $B \in \mathcal{B}_x$ such that $B \subset U$. \mathcal{B} is called the **topological basis** of X if for any $x \in X$ there exists $\mathcal{B}_x \subset \mathcal{B}$ that is a local bases in x .

Hence \mathcal{B} is a topological basis of X iff for any $x \in X$ and for any neighborhood U of x there exists $B \in \mathcal{B}_x$ such that $x \in B$ and $B \subset U$.

Exercise 7.0.5 Show that the collection of open balls is the topological basis of n dimensional Euclidean space \mathbb{R}^n .

If \mathcal{B} is a topological basis of a topological space X then it satisfies the following: (i) For any $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$; and (ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in (B_1 \cap B_2)$ there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset (B_1 \cap B_2)$. Assume now that we have a set X (without a topology) and a collection \mathcal{B} of subsets of X satisfying conditions (i) and (ii). We say that $S \subset X$ is open if for any $x \in S$ there exists $B \in \mathcal{B}$ such that $x \in B \subset S$. The collection of all these open sets is a topology on X .

Exercise 7.0.6 Prove the above statement.

The method of defining a topology on X through a basis is very important. This method was used in Chapter 3 in defining a topology on \mathbb{R} where the basis \mathcal{B} was the collection of all open intervals.

Definition 592 Let X be a topological space with a topology \mathcal{O} and let $X_0 \subset X$. Then we can define a topology \mathcal{O}_0 on X_0 as the collection of all sets of the form $O \cap X_0$ where $O \in \mathcal{O}$. \mathcal{O}_0 is called the **relative topology** on X_0 created by \mathcal{O} and X_0 is called a **topological subspace** of X .

Example 593 Let $X = \mathbb{R}$ be a topological space with the usual topology in Definition 104. Let $X_0 = [0, 1)$. Then the relative topology \mathcal{O}_0 on X_0 is the collection of all sets of the form $O \cap [0, 1)$ where O is open in \mathbb{R} . For example $O_0 = [0, \frac{1}{2})$ is open in X_0 because $[0, \frac{1}{2}) = (-1, 1) \cap [0, \frac{1}{2})$ and $(-1, 1)$ is open in \mathbb{R} .

7.1 Continuous Functions and Homeomorphisms

Let X, Y be topological spaces and $f : X \rightarrow Y$ be a function from X to Y . We say that f is continuous at $x_0 \in X$ if for any neighborhood V of $f(x_0)$ in Y the inverse image $f^{-1}(V)$ is a neighborhood of x_0 in X . f is continuous on X if f is continuous at every $x \in X$. This is similar to Definition 244 where neighborhood has been substituted for open ball.

Exercise 7.1.1 Prove that $f : X \rightarrow Y$ is continuous iff $f^{-1}(V)$ is open (closed) in X for any V open (closed) in Y .

Definition 594 Let $f : X \rightarrow Y$ be a function from X to Y . Assume that there exists an inverse function $f^{-1} : Y \rightarrow X$ and let both f and f^{-1} be continuous. Then we say that f is a **homeomorphism** of X onto Y and that X and Y are **homeomorphic**. Homeomorphic means topologically equivalent (i.e. the same from a topological point of view).

Example 595 Let $X = \mathbb{R}$ with a topology determined by the Euclidean metric $d_2(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Let $Y = \mathbb{R}$ with a topology determined by the sup metric $d_\infty(x, y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Then X and Y are homeomorphic. Hence from a topological point of view a circle and a square are indistinguishable.

We could define other topological properties like compactness, connectedness, and separability but we will only touch upon them. All these notions are defined in Chapter 4. Notice that although they are defined in a metric space, these definitions don't use the notion of distance (they are simply formulated in terms of open sets).

7.2 Separation Axioms

Since the notion of distance is very natural for us, a metric space is more easily envisioned than a topological space. That is why we take for granted many results. For instance, in a metric space given two different elements $x, y \in X$, $x \neq y$ there exists two disjoint open sets U, V each containing just one element (i.e. $x \in U$, $y \in V$, and $U \cap V = \emptyset$). See Figure 7.3. But this is not necessarily true in a general topological space. Before we show this we will state separation axioms

Definition 596 A topological space X is called: (i) a **T_0 -space** if for any two distinct elements x, y there exists a neighborhood U of x not containing y ; (ii) a **T_1 -space** if for any two distinct elements x, y there exists a neighborhood U of x not containing y and a neighborhood V of y not containing x ; (iii) a **T_2 -space (or Hausdorff space)** if any two distinct elements x, y have disjoint neighborhoods (i.e. there exist two open sets U, V such that $x \in U$,

$y \in V$ and $U \cap V = \emptyset$); (iv) a T_3 -**space** if any closed set A and an element $x \notin A$ have disjoint neighborhoods (i.e. any point x and any closed set A not containing x can be separated by disjoint open sets); (v) a T_4 -**space** if any two disjoint closed sets have two disjoint neighborhoods (i.e. any two disjoint closed sets can be separated by disjoint open sets). These axioms can be pictured in Figure 7.4. A **regular space** is a T_3 -space which is also a T_1 -space. A **normal space** is a T_4 -space which is also a T_1 -space.

Note that there may be slightly different terminology in the literature depending on the book you reference.

Exercise 7.2.1 Show that the following sequence of implications holds true: Normal space \implies regular space \implies Hausdorff space \implies T_1 -space \implies T_0 -space.

Exercise 7.2.2 Show that any metric space is normal. Hint: A positive distance can be always bisected.

Combining the statements of Exercises 7.2.1 and 7.2.2 we get that any metric space satisfies all the separation axioms. Now we are going to show that none of the implications of Exercise 7.2.1 can be reversed.

Example 597 A set X containing at least two distinct elements with the trivial the topology (defined in the Example 589) is not a T_0 -space. To see this, let $x \neq y$. x cannot be separated by an open set from y because the only open set containing x is the whole set X (that also contains y).

Before giving other examples it is useful to state the following theorem that you should prove as an exercise.

Exercise 7.2.3 A topological space X is T_1 iff every singleton is closed.

Example 598 Let $X = \{a, b\}$ and $\mathcal{O} = \{\emptyset, \{a\}, X\}$ be a topology on X . Note that by definition, $\{a\}$ is open. To show that \mathcal{O} is a topology we must satisfy the conditions in Definition 585. Obviously $\emptyset, X \in \mathcal{O}$ by construction. On closedness with respect to arbitrary union, $\{a\} \cup \emptyset = \{a\} \in \mathcal{O}$ and $\{a\} \cup X = X \in \mathcal{O}$. On closedness with respect to finite intersection $\{a\} \cap X = \{a\} \in \mathcal{O}$ and $\{a\} \cap \emptyset = \emptyset \in \mathcal{O}$. Now \mathcal{O} is a T_0 -space because for $a \neq b$, the open set $\{a\}$ is a neighborhood of the element a not containing b . Note that $\{b\}$ is closed. According to Exercise 7.2.3 \mathcal{O} is not a T_1 -space because $\{a\}$ is not closed.

Example 599 Let $X = \mathbb{N}$ with the topology of finite complements (defined in Example 590). This is a T_1 -space because $\{x\}$ is closed for $x \in \mathbb{N}$ (note $\mathbb{N} \setminus \{x\}$ is infinite thus open). It is not Hausdorff since: an open set A containing 1 has the form $A = \mathbb{N} \setminus \{x_1, \dots, x_m\}$ where $x_i \neq 1$; an open set B containing 2 has the form $B = \mathbb{N} \setminus \{y_1, \dots, y_n\}$ where $y_i \neq 2$; and the sets A, B are not disjoint.

Example 600 Let $X = \mathbb{R}$ and the topology \mathcal{O} consists of: (i) all open sets in the usual Euclidean topology (i.e. the topology induced by the Euclidean metric); and (ii) all sets of the form $U \setminus K$ where $K = \{\frac{1}{n}, n \in \mathbb{N}\}$ and U is open in the usual Euclidean topology. This is a Hausdorff space because open sets of type (i) can be used for separating two distinct points. It is not a T_3 -space because K is closed and $0 \notin K$ cannot be separated from K .

None of the topological spaces in Examples 597 to 600 are metrizable. Why? Let (X, \mathcal{O}) be a topological space which is metrizable by d . Then (X, d) is a metric space. That is, let \mathcal{O}' be the collection of all open sets of (X, d) . Then \mathcal{O}' coincides with \mathcal{O} (ie. this means that (X, \mathcal{O}) is metrizable). Then (X, \mathcal{O}) and (X, \mathcal{O}') are identical topological spaces one of which is not T_0 and the other is normal (because it is a metric space by Exercise 7.2.2). Hence, there is a contradiction. The same argument can be used in the other examples. Hence being a normal topological space is a necessary condition for the space to be metrizable. But this condition is not sufficient. For further reading see Kelley (???)

7.3 Convergence and Completeness

In Section 4.1 the notion of the convergence of a sequence in a metric space was defined. In Definition 143 we characterize a closed set A as the set containing limit points of all convergent sequences from A . Defining closed sets we can also define open sets as their complements. This means we can define a topology. That is, a topology in a metric space can be defined in terms of convergence of a sequence (as we did in Chapter 4). Can this procedure be used in a topological space? First, we must address whether convergence of a sequence can be introduced in topological space? In Definition 136 we see that the concept of distance (metric) is used there (we say that $\langle x_n \rangle \rightarrow x$ if for any $\varepsilon > 0$, $\exists N$ such that $\forall n \geq N$, $d(x_n, x) < \varepsilon$). In this definition, ε represents an ε -ball around x (i.e. a neighborhood of x). Hence the definition

can be reformulated as the following: $\langle x_n \rangle \longrightarrow x$ if for any neighborhood U of x there exists N such that $\forall n \geq N, x_n \in U$. In this new version only topological notions are used and hence convergence of a sequence can be defined in a topological space.

You may wonder if a topology can be built only in terms of convergence of sequences (the same way it is done in a metric space). The answer is not always. Loosely speaking it is possible in topological spaces that are separable (i.e. containing a countably dense set). Thus separability of a topological (metric) space is an important property.

Exercise 7.3.1 *Show that if X has a countable basis then it is separable.*

Among topological spaces that are not separable there exist spaces whose topology cannot be fully built only in terms of convergence of sequences. If we want to build a topology in these spaces in terms of convergence, then the notion of sequence has to be replaced by the more general notion of a net. We will not deal with it here (again see Kelley).

The last important property of a metric space is completeness. Is this property topological? That is, can completeness be defined in terms of open sets? As we know, defining completeness requires the notion of a Cauchy sequence and Definition 169 of a Cauchy sequence is based on the concept of distance. It cannot be defined without a metric. In other words, a Cauchy sequence cannot be defined in a general topological space. Hence completeness is not a topological property as the next example shows.

Example 601 *Let $X = (0, 1], |\cdot|$, $Y = ([1, \infty), |\cdot|)$ be two metric spaces. Then $f : X \rightarrow Y$ given by $f(x) = \frac{1}{x}$ is a homeomorphism (f is a bijection and f and f^{-1} are continuous). Hence these two metric space are topologically equivalent but X is not complete whereas Y is.*

Total boundedness is not a topological property either. Example 601 shows this since X is totally bounded whereas Y is not. Theorem 198 says that compactness in a metric space is equivalent to completeness and total boundedness. Compactness is a topological property while completeness and total boundedness are not topological properties individually but if they occur simultaneously they are a topological property.